On certain Hamiltonian systems related to the cubic Szegö equation

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Résumé

Cette thèse est principalement consacrée à l'étude du comportement en temps long de solutions de certaines équations aux dérivées partielles hamiltoniennes, du type

$$i\partial_t u = X_H(u) \ ,$$

en particulier l'existence globale, la croissance des normes de Sobolev, la diffusion et l'approximation par la dynamique résonante.

Dans ce contexte, nous considérons d'abord une perturbation de l'équation de Szegő cubique par un potentiel linéaire,

$$i\partial_t u = \Pi_+(|u|^2 u) + \alpha \int_{\mathbb{S}^1} u \ , \ \alpha \in \mathbb{R} \ , \tag{α–Szegő}$$

où Π_+ désigne le projecteur de Szegő sur les fréquences positives. Pour $\alpha = 0$, cette équation est l'équation de Szegő cubique, étudiée récemment par Gérard et Grellier comme modèle mathématique d'équation non linéaire et non dispersive.

Pour l'équation (α–Szegő), nous établissons le caractère bien posé et la complète intégrabilité, et étudions la dynamique des valeurs singulières des opérateurs de Hankel associés. En outre, nous montrons les propriétés suivantes pour cette équation, sur une classe de sous–variétés invariantes de dimensions finies arbitrairement grandes : si $\alpha < 0$, toute trajectoire est relativement compacte, et toute norme de Sobolev est bornée le long de cette trajectoire. Si $\alpha > 0$, il existe des trajectoires le long desquelles toutes les normes de Sobolev de régularité plus grande que $\frac{1}{2}$ tendent exponentiellement vers l'infini en temps.

Dans une seconde partie, nous étudions un système mixte Schrödinger–ondes sur le cylindre $(x, y) \in \mathbb{R} \times \mathbb{T}$,

$$i\partial_t U + \partial_{xx} U - |D_y| U = |U|^2 U \ . \tag{WS}$$

En adaptant une idée de Hani–Pausader–Tzvetkov–Visciglia, nous établissons une théorie du scattering modifiée reliant les petites solutions de cette équation et les petites solutions de l'équation de Szegő cubique. En combinant cette théorie du scattering avec un résultat récent de Gérard–Grellier, nous en déduisons l'existence de solutions globales de (WS) qui sont non bornées dans l'espace $L^2_x H^s_y(\mathbb{R} \times \mathbb{T})$ pour tout $s > \frac{1}{2}$.

Mots-clés : équation de Szegő, paire de Lax, systèmes hamiltoniens intégrables, équation mixte Schrödinger–ondes, scattering modifié, cascade d'énergie, explosion en grand temps, turbulence faible.

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Abstract

The main purpose of this ph.D. thesis is to study the long time behavior of solutions to some Hamiltonian PDEs,

$$i\partial_t u = X_H(u) \ ,$$

including global existence, growth of high Sobolev norms, scattering and long time approximation by resonant dynamics.

In this context, at first we consider the Szegő equation on the circle $\mathbb{S}^1$ perturbed by a linear potential,

$$i\partial_t u = \Pi_+(|u|^2 u) + \alpha \int_{\mathbb{S}^1} u \ , \ \alpha \in \mathbb{R} \ , \tag{α–Szegő}$$

where Π_+ is the projector onto the non-negative frequencies. For $\alpha = 0$, it turns out to be the cubic Szegő equation, which was recently introduced by Gérard and Grellier as a mathematical toy model of a non-linear totally non dispersive equation.

We study the global well-posedness, the integrability and the dynamics of the singular values of the related Hankel operators of the α–Szegő equation. Moreover, we establish the following properties for this equation on a class of invariant submanifolds, with an arbitrary large dimension. For $\alpha < 0$, any trajectory is relatively compact, and all the Sobolev norms are bounded on it. For $\alpha > 0$, there exist trajectories on which every Sobolev norm of regularity $s > \frac{1}{2}$ exponentially tends to infinity in time.

Second, we study the wave-guide Schrödinger equation posed on the spatial domain $(x, y) \in \mathbb{R} \times \mathbb{T}$,

$$i\partial_t U + \partial_{xx} U - |D_y| U = |U|^2 U \ . \tag{WS}$$

Adapting an idea by Hani–Pausader–Tzvetkov–Visciglia, we establish a modified scattering theory between small solutions to this equation and small solutions to the cubic Szegő equation. Combining this scattering theory with a recent result by Gérard–Grellier, we infer existence of global solutions to (WS) which are unbounded in the space $L^2_x H^s_y(\mathbb{R}\times\mathbb{T})$ for every $s > \frac{1}{2}$.

Keywords: Szegő equation, Lax pair, integrable Hamiltonian systems, wave guide Schrödinger equation, modified scattering, energy cascade, large time blow up, weak turbulence.

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Introduction générale

L'étude du comportement en grand temps des solutions d'équations hamiltoniennes de type Schrödinger non linéaire est une question centrale dans la théorie des équations aux dérivées partielles non linéaires dispersives. En particulier, lorsque les solutions sont définies pour tout temps, un problème naturel est de déterminer si elles deviennent ou non très oscillantes en grand temps, c'est–à–dire si une part suffisamment importante de l'énergie est transférée vers les petites échelles spatiales. La théorie physique s'attachant à décrire un tel phénomène est appelée turbulence faible — ou turbulence d'ondes. Initialement, le mot latin *turba* désigne la foule, et le mot turbulence désigne étymologiquement le désordre causé par le mouvement de grandes populations. Léonard de Vinci fut probablement le premier à utiliser le terme au sens du mouvement des fluides. La terminologie de turbulence est maintenant couramment utilisée pour désigner le mouvement désordonné des fluides — par opposition au mouvement laminaire — et cette théorie a connu une impulsion importante au milieu du vingtième siècle avec un célèbre article de Kolmogorov [34]. À la suite de Kolmogorov, la théorie de la turbulence faible a été développée dans les années soixante pour décrire les régimes dynamiques d'ondes non linéaires dans lesquels de l'énergie est transférée entre grandes et petites échelles [1, 2, 29, 33, 36, 37, 51].

Du point de vue mathématique, le cadre naturel est celui des équations aux dérivées partielles non linéaires hamiltoniennes bien posées globalement en temps. Un exemple typique est fourni par la classe des équations de Schrödinger non linéaires sur une variété, par exemple l'espace euclidien $\mathbb{R}^n$ ou le tore $\mathbb{T}^n$, dans les régimes défocalisants et sous–critiques. Dans ce cadre, on parlera de turbulence faible si les solutions considérées ont des normes Sobolev non bornées en grand temps. Malheureusement, pour l'équation de Schrödinger, ce phénomène est très difficile à mettre en évidence, et encore plus à décrire — voir le paragraphe 0.1 ci–dessous.

Le but de cette thèse est mettre en évidence le phénomène de turbulence faible pour une classe d'évolutions hamiltoniennes qui sont des modèles simples d'interactions d'ondes non linéaires, et sont reliés à un système intégrable particulier, l'équation de Szegő cubique. Dans cette introduction, nous rappelons d'abord l'état des connaissances sur la turbulence faible pour les équations de Schrödinger non linéaires. Puis nous décrivons l'équation de Szegő cubique sur le cercle, introduite récemment par P. Gérard et S. Grellier, et les résultats de turbulence faible qu'ils ont obtenus. Enfin, nous présentons les deux équations qui font l'objet de cette thèse, pour lesquelles nous étudions l'existence de solutions turbulentes : une perturbation de l'équation de Szegő cubique sur le cercle, et un modèle mixte Schrödinger-ondes cubique sur le cylindre.

Terminons ce premier paragraphe en donnant une définition précise de ce que nous appellerons solution turbulente.

Définition 0.0.1 (Turbulence faible pour un système hamiltonien)**.** *Soit une équation aux dérivées partielles hamiltonienne*

$$\partial_t u = X_H(u) \ ,$$

globalement bien posée sur $\cap_{s>0} H^s(M)$, où M est une variété. On dit qu'une solution u est turbulente si, pour s assez grand,

$$\limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty \ . \tag{0.0.1}$$

0.1 Le cas des équations de Schrödinger on linéaires

Soit M une variété riemannienne de dimension $1, 2$ ou 3. On suppose soit que M est compacte, soit que la métrique sur M est convenablement contrôlée à l'infini, par exemple que M est le produit cartésien de $\mathbb{R}^p$ avec une variété riemannienne compacte. On considère l'équation de Schrödinger non linéaire cubique,

$$\begin{cases} i\partial_t u - \Delta u + |u|^2 u = 0 \ , \\ u(0,x) = u_0(x) \ , \end{cases} \tag{NLS}$$

où u est une fonction à valeurs complexes avec la variable spatiale $x \in M$. On sait [5], que cette équation est globalement bien posée sur $H^s(M)$ pour tout $s > 1$. De plus, les lois de conservation de la masse et de l'énergie

$$\|u(t)\|_{L^2}^2 = \|u(0)\|_{L^2}^2 \ , \ \int_M (|\nabla u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4)\, dx = E$$

assurent que la norme H^1 de toute solution est bornée au cours du temps. En utilisant des estimées dispersives, Bourgain [3] et Staffilani [47] ont prouvé, dans le cas où M est un tore, que

$$\|u(t)\|_{H^s} \lesssim t^{C(s-1)} \|u(0)\|_{H^s} \ . \tag{0.1.1}$$

Toutefois, comme on va le voir, on est très loin de savoir si ces estimations sont optimales ou non.

Dans le cas où M est de dimension 1, l'équation (NLS) est complètement intégrable [52, 20], et les lois de conservation contrôlent toute la régularité, de sorte que pour $s > 1$,

$$\|u(t,\cdot)\|_{H^s(\mathbb{T}^1)} \leq C(\|u_0\|_{H^s(\mathbb{T}^1)}) \ , \forall t \in \mathbb{R} \ .$$

Dans le cas où $M = \mathbb{R}^d$ pour $d = 2, 3$, les résultats de scattering de Ginibre-Velo [19] et Dodson [8] permettent aussi de montrer que les normes H^s de la solution restent uniformément bornées.

Dans le cas où M est un tore de dimension supérieure à 1, le problème de l'existence de solutions turbulentes au sens de la définition 0.0.1 a été posé par Bourgain dans [4]. Ce problème est toujours ouvert, malgré plusieurs progrès dans cette direction, que nous rappelons maintenant.

Colliander–Keel–Staffilani–Takaoka–Tao ont construit des solutions de l'équation (NLS), admettant une petite norme H^s à l'instant initial, et une grande norme H^s à un instant T avec $s > 1$.

Théorème 0.1.1 ([7]). *Pour tous $\varepsilon > 0$, $K > 0$, $s > 1$, il existe une solution lisse $u(t,x)$ de l'équation (NLS) et un temps $T > 0$ tels que*

$$\|u(0)\|_{H^s(\mathbb{T}^2)} \leq \varepsilon \text{ et } \|u(T)\|_{H^s(\mathbb{T}^2)} > K \ .$$

En précisant cette construction, Guardia et Kaloshin améliorent les résultats et estiment la vitesse de transition vers les hautes fréquences.

Théorème 0.1.2 ([24]). *Soient $s > 1$, il existe une constante $c > 0$ avec la propriété suivante :*
Pour $K \geq 1$, il existe une solution globale $u(t,x)$ of (NLS) et un grand temps T vérifiant $0 < T \leq K^c$, telles que

$$\|u(T)\|_{H^s} \geq K\|u(0)\|_{H^s} \ .$$

En outre, cette solution peut être choisie de sorte que

$$\|u(0)\|_{L^2} \leq K^{-(s-1)c/4+2/(s-1)} \ .$$

Ce résultat a été généralisé pour l'équation de Schrödinger non linéaire cubique avec un potentiel de convolution [22]. Ces techniques ont également été appliquées à l'équation de Schrödinger avec d'autres non–linéarités [31, 30, 23].

On notera que les résultats ci–dessus sont plus faibles qu'un résultat de turbulence faible, puisqu'ils ne précisent pas si la trajectoire considérée est ou non bornée dans H^s. La démonstration de ces résultats est fondée sur l'approximation sur un temps assez long par la solution de la forme totalement résonante associée à l'équation (NLS) sur $\mathbb{T}^2$. Il était dès lors naturel d'approfondir l'étude des solutions de cette forme normale elle–même. C'est l'objet du travail de Hani [25], qui a prouvé l'existence d'orbites de Sobolev non bornées pour une famille de non linéarités hamiltoniennes cubiques qui inclut cette forme totalement résonante. Hélas, ce résultat n'entraîne pas de résultat analogue pour (NLS) sur $\mathbb{T}^2$.

Plus récemment, Hani–Pausader–Tzvetkov–Visciglia ont considéré (NLS) dans le domaine spatial $M = \mathbb{R} \times \mathbb{T}^d$ $(1 \leq d \leq 4)$ [26]. Ces auteurs ont montré que la dynamique asymptotique des petites solutions est liée par un opérateur de scattering modifié à celle des petites solutions du système résonant

$$\begin{aligned} i\partial_\tau G(\tau) = &\quad \mathcal{R}[G(\tau), G(\tau), G(\tau)], \\ \mathcal{F}_{\mathbb{R}\times\mathbb{T}^d}\, \mathcal{R}[G,G,G](\xi,p) = &\sum_{\substack{p_1+p_3=p+p_2 \\ |p_1|^2+|p_3|^2=|p|^2+|p_2|^2}} \widehat{G}(\xi,p_1)\overline{\widehat{G}(\xi,p_2)}\widehat{G}(\xi,p_3) \ , \end{aligned} \qquad (0.1.2)$$

où $\widehat{G}(\xi,p)=\mathcal{F}_{\mathbb{R}\times\mathbb{T}^d}G(\xi,p)$ est la transformation de Fourier de G au point $(\xi,p)\in\mathbb{R}\times\mathbb{Z}^d$. Le système ci–dessus est précisément le système totalement résonant associé à l'équation de Schrödinger sur $\mathbb{T}^2$, si l'on considère ξ comme un paramètre. Quand $d\geq 2$, le résultat de [25] montre qu'il existe des orbites de Sobolev non bornées pour (0.1.2) . On en déduit l'existence de solutions turbulentes sur $\mathbb{R}\times\mathbb{T}^2$.

Théorème 0.1.3 ([26]). *Soit $s\in\mathbb{N}$, $s\geq 30$. Alors, pour tout $\varepsilon>0$, il existe une solution globale $U(t)$ de l'équation (NLS) sur $\mathbb{R}\times\mathbb{T}^2$, telle que*

$$\|U(0)\|_{H^s(\mathbb{R}\times\mathbb{T}^2)}\leq\varepsilon,\qquad \limsup_{t\to+\infty}\|U(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^2)}=+\infty\ .$$

Plus précisément, il existe une suite (t_k) tendant vers l'infini et $c>0$ tels que

$$\|U(t_k)\|_{H^s(\mathbb{R}\times\mathbb{T}^2)}\geq c\exp(c(\log\log\log t_k)^{\frac{1}{2}})\ .$$

À notre connaissance, ce dernier résultat est actuellement le seul résultat d'existence de solution turbulente pour l'équation (NLS).

0.2 L'équation de Szegő cubique

L'équation de Szegő cubique s'écrit

$$\begin{cases} i\partial_t u=\Pi_+(|u|^2u)\ ,\ (t,x)\in\mathbb{R}\times\mathbb{S}^1\\ u(0)=u_0\ ,\end{cases}\tag{Szegő}$$

où Π_+ est le projecteur de Szegő sur les fréquences positives ou nulles. Elle a été récemment introduite et étudiée par Gérard et Grellier dans [9, 11, 13, 16] comme un modèle mathématique d'une équation hamiltonienne non linéaire, totalement non dispersive, complètement intégrable. On note $\Pi_-:=\mathrm{Id}-\Pi_+$ et $u_\pm:=\Pi_\pm u$.

Un exemple qui motive l'introduction de l'equation de Szegő cubique est l'équation de demi–onde non linéaire suivante :

$$i\partial_t u-|D|u=|u|^2u\ ,\ |D|=\sqrt{-\partial_{xx}}\ ,\ x\in\mathbb{S}^1\ .\tag{HW}$$

C'est en effet une équation des ondes non linéaire car si l'on applique l'opérateur $i\partial_t+|D|$ aux deux membres de l'équation, on obtient

$$-\partial_{tt}u+\partial_{xx}u=|u|^4u+2|u|^2(|D|u)-u^2(|D|\overline{u})+|D|(|u|^2u)\ .$$

On remarque tout d'abord que l'équation (HW) est non dispersive parce que, en la projetant sur des fréquences positives/négatives, on obtient le système suivant d'équations de transport :

$$\begin{cases} i(\partial_t u_+ +\partial_x u_+)=\Pi_+(|u|^2u)\\ i(\partial_t u_- -\partial_x u_-)=\Pi_-(|u|^2u)\ .\end{cases}$$

De plus, si la donnée initiale u_0 satisfait $u_0 = \Pi_+(u_0) \in H^s(\mathbb{S}^1)$ avec une petite norme ε, alors la solution correspondante u est approchée, sur des intervalles de temps assez longs — de l'ordre de $\varepsilon^{-2}|\ln \varepsilon|$ — par la solution v de l'équation suivante [12],

$$i(\partial_t v + \partial_x v) = \Pi_+(|v|^2 v) \ .$$

Notons qu'un changement de variable élémentaire réduit cette équation à l'équation (Szegő). L'équation de Szegő cubique est en fait la forme totalement résonante de l'équation de demi–onde (HW), au même titre que l'équation (0.1.2) pour l'équation de Schrödinger cubique. En effet, l'ensemble résonant de l'équation de demi–onde est donné par

$$\begin{aligned}\Gamma :&= \{(p_1, p_2, p_3, p_4) \in \mathbb{Z}^4 : \quad p_1 + p_3 = p_2 + p_4 \ , \ |p_1| + |p_3| = |p_2| + |p_4|\} \\ &= \{\ p_1 = p_4,\ p_2 = p_3\} \cup \{\ p_1 = p_2,\ p_3 = p_4\} \cup \{\ p_j \geq 0 \ , \ \forall j\} \cup \{\ p_j \leq 0 \ , \ \forall j\} \ ,\end{aligned}$$

ce qui permet de se ramener à (Szegő) [12].

L'équation de Szegő cubique a été introduite et étudiée dans [9, 11, 13, 16] sur $\mathbb{S}^1$. On présente les principaux résultats ci–dessous. D'abord, pour tout sous-espace X de $\mathcal{D}'(\mathbb{S}^1)$, on note

$$X_+ := \{u \in X : \ \Pi_+(u) = u\} \ .$$

Ainsi, L^2_+ est l'espace de Hardy usuel des fonctions holomorphes dans le disque unité dont les traces sont L^2 sur le cercle unité. L'équation de Szegő cubique (Szegő) décrit le flot hamiltonien d'énergie

$$E(u) = \frac{1}{4}\int_{\mathbb{S}^1} |u|^4 \frac{d\theta}{2\pi} \ ,$$

pour la forme symplectique

$$\omega(u, v) = \operatorname{Im} \int_{\mathbb{S}^1} u\overline{v}\, \frac{d\theta}{2\pi} \ .$$

De cette structure hamiltonienne, on déduit une première loi de conservation,

$$E(u) = E(u_0) \ .$$

De plus, l'invariance par la multiplication de u par des nombres complexes de module 1 et l'invariance par translation sur $\mathbb{S}^1$ induit deux lois de conservation,

$$Q(u) := \int_{\mathbb{S}^1} |u|^2 \frac{d\theta}{2\pi} \ , \ M(u) := (Du|u) \ , \ D := -i\partial_\theta \ .$$

L'étude du problème de Cauchy est résumée ci–dessous.

Théorème 0.2.1 ([11])**.** *Pour tout $u_0 \in H^{\frac{1}{2}}_+(\mathbb{S}^1)$, il existe une unique solution $u \in C(\mathbb{R}, H^{\frac{1}{2}}_+(\mathbb{S}^1))$ de (Szegő) telle que $u(0) = u_0$. De plus, si $u_0 \in H^s_+(\mathbb{S}^1)$ pour un $s > \frac{1}{2}$, alors $u \in C^\infty(\mathbb{R}, H^s_+(\mathbb{S}^1))$.*

Un fait remarquable est que l'équation (Szegő) admet beaucoup plus de lois de conservation, et qu'elle peut être approchée par une suite de systèmes hamiltoniens de dimensions finies complètement intégrables au sens de Liouville. C'est une conséquence d'une structure de paire de Lax que nous allons maintenant décrire. Pour cela, nous définissons les opérateurs de Hankel et les opérateurs de Toeplitz sur le cercle.

Définition 0.2.1. *Pour toute fonction $b \in L^\infty(\mathbb{S}^1)$, on définit l'opérateur de Hankel $H_b : L^2_+ \to L^2_+$ par*

$$H_b(h) := \Pi_+(b\overline{h}) \ .$$

L'opérateur de Toeplitz de symbole $b \in L^\infty(\mathbb{S}^1)$ est donné par

$$T_b(h) := \Pi_+(uh) \ .$$

Gérard et Grellier ont trouvé deux paires de Lax pour des opérateurs de Hankel liés à la solution de (Szegő).

Théorème 0.2.2. *Toute solution $u \in C(\mathbb{R}, H^s_+)$, $s > \frac{1}{2}$ de l'équation de Szegő cubique,*

$$i\partial_t u = \Pi_+(|u|^2 u) \ ,$$

vérifie les identités

$$\frac{d}{dt}H_u = [B_u, H_u] \ , \ \frac{d}{dt}K_u = [C_u, K_u] \ , \tag{0.2.1}$$

avec $B_u = \frac{i}{2}H_u^2 - iT_{|u|^2}$, $K_u = T_z^* H_u$, $C_u = \frac{i}{2}K_u^2 - iT_{|u|^2}$. En d'autres termes, (H_u, B_u) et (K_u, C_u) sont deux paires de Lax pour l'équation de Szegő cubique.

Un corollaire est que la famille des opérateurs autoadjoints compacts $(H^2_{u(t)})_{t\in\mathbb{R}}$ est isospectrale à $H^2_{u_0}$. En utilisant les résultats de Nehari [35] et Peller [41], il s'ensuit que les normes $\mathrm{BMO}(\mathbb{S}^1)$ et $B^1_{1,1}(\mathbb{S}^1))$, respectivement équivalentes à la norme d'opérateur et à la norme trace de H_u sur L^2_+, sont essentiellement conservées. En particulier, en utilisant de plus le lemme de Gronwall, on obtient les estimations *a priori* suivantes.

Corollaire 0.2.1. *Si $u_0 \in H^s_+$ pour un $s > 1$, la solution $u(t)$ correspondante de* (Szegő) *satisfait à*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{L^\infty} \le C_s \|u_0\|_{H^s} \ ,$$
$$\|u(t)\|_{H^s} \le C_s \|u_0\|_{H^s} \mathrm{e}^{C_s\|u_0\|^2_{H^s}|t|} \ .$$

Un autre corollaire est l'existence de variétés de dimension finie invariantes. Pour tout entier $d \ge 1$, on désigne par $\mathcal{V}(d)$ l'ensemble de symboles u tels que $\mathrm{rk}H_u + \mathrm{rk}K_u = d$ (où $\mathrm{rk}T$ désigne le rang de l'opérateur T). Le théorème de Kronecker assure que $\mathcal{V}(d)$ coïncide avec la variété des fractions rationnelles de la forme

$$u(z) = \frac{A(z)}{B(z)} \ , \ z \in \mathbb{S}^1 \ ,$$

où A et B sont deux polynômes et sont premiers entre eux, les zéros de B sont tous de module > 1, et

– si $d = 2N$ est un nombre pair, $A \in \mathbb{C}_{N-1}$, $B \in \mathbb{C}_N$, $\deg(A) \leq N-1$ et $\deg(B) = N$,
– si $d = 2N - 1$ est un nombre impair, $A \in \mathbb{C}_{N-1}$ et $B \in \mathbb{C}_{N-1}$, $\deg(A) = N - 1$ et $\deg(B) \leq N - 1$.

Le flot de Szegő cubique est complètement intégrable sur $\mathcal{V}(d)$ [11]. En effet, pour tout entier n, posons

$$J_n(u) := (H_u^n(1)|1) \ .$$

Alors J_n est une loi de conservation pour tout entier n pair. Toutes les lois de conservation J_{2p} sont génériquement indépendantes sur $\mathcal{V}(d)$ et satisfont à la relation

$$\{J_{2p}, J_{2q}\} = 0 \ .$$

Pour toute donnée initiale dans $\mathcal{V}(d)$, les solutions de (Szegő) sont quasi-périodiques [16]. Pour tout $u_0 \in H_+^{\frac{1}{2}}(\mathbb{S}^1)$, la trajectoire

$$t \in \mathbb{R} \to u(t) \in H_+^{\frac{1}{2}}(\mathbb{S}^1) \text{ solution de (Szegő)}$$

est presque périodique [15] .

La démonstration de ces résultats est fondée sur la construction [15] d'une transformation de Fourier non linéaire sur $H_+^{\frac{1}{2}}(\mathbb{S}^1)$, que nous allons décrire brièvement.

D'abord, nous introduisons des notations supplémentaires. Pour tout entier $d \geq 0$, nous rappelons qu'un produit de Blaschke de degré d est une fonction rationnelle sous la forme

$$\Psi(z) = \mathrm{e}^{-i\psi} \prod_{j=1}^{d} \frac{z - p_j}{1 - \overline{p_j} z} \ , \ \psi \in \mathbb{T} \ , \ p_j \in \mathbb{D} \ ,$$

où ψ est appelée l'angle de Ψ. On note $\mathcal{B}_d$ l'ensemble des produits de Blaschke de degré d.

Pour tout $\tau \geq 0$, on définit

$$E_u(\tau) := \ker(H_u^2 - \tau^2 \mathrm{I}), \ F_u(\tau) := \ker(K_u^2 - \tau^2 \mathrm{I}) \ . \tag{0.2.2}$$

On définit aussi les ensembles singuliers dominants de H_u et K_u,

$$\Sigma_H(u) := \{\tau > 0 : \ u \not\perp E_u(\tau)\}, \quad \Sigma_K(u) := \{\tau \geq 0 : \ u \not\perp F_u(\tau)\} \ .$$

Notons que les éléments de ces deux ensembles sont alternés à cause du principe du min-max. Et nous appelons les éléments de $\rho_j \in \Sigma_H(u)$ et $\sigma_k \in \Sigma_K(u)$ les valeurs singulières H-dominantes et K-dominantes respectivement.

Le symbole u se décompose en

$$u = \sum_{\{j: \ \rho_j \in \Sigma_H(u)\}} u_j = \sum_{\{k: \ \sigma_k \in \Sigma_K(u)\}} u'_k \ ,$$

et chaque composante u_j est la projection de u sur l'espace $E_u(\rho_j)$ et u'_k est la projection sur $F_u(\sigma_k)$. De plus, il existe $\Psi_{2j-1} \in \mathcal{B}_\ell$ avec $\ell = \dim E_u(\rho_j) - 1$ et $\Psi_{2k} \in \mathcal{B}_m$ avec $m = \dim F_u(\sigma_k) - 1$, tel que

$$H_u(u_j) = \Psi_{2j-1}^{-1} \rho_j u_j \ , \ K_u(u'_k) = \Psi_{2k} \sigma_k u'_k \ . \tag{0.2.3}$$

Si le cardinal de $\Sigma_H(u) = \Sigma_K(u)$ est fini égal à N, la fonction u s'écrit explicitement

$$u(z) = \langle \mathscr{C}(z)^{-1}(\Psi_{\text{odd}}(z)), 1_N \rangle,$$

où $\mathscr{C}(z)$ est la matrice $N \times N$ dont les coefficients sont

$$c_{jk}(z) := \frac{\rho_j - \sigma_k z \Psi_{2k}(z)\Psi_{2j-1}(z)}{\rho_j^2 - \sigma_k^2} \ , \ 1 \leq j, k \leq N \ ,$$

$\Psi_{\text{odd}}(z)$ est le vecteur colonne constitués des $\Psi_{2j-1}(z)$ pour $1 \leq j \leq N$, et 1_N est le vecteur colonne de taille N dont toutes les composantes sont égales à 1.

Si l'on fixe les zéros de chaque produit de Blaschke Ψ_r et si l'on fait varier les angles ψ_r et les valeurs singulières $(s_1, \dots, s_d)$ définies par

$$s_{2j-1} = \rho_j, \rho_j \in \Sigma_H(u) \ , \ s_{2k} = \sigma_k, \sigma_k \in \Sigma_K(u) \setminus \{0\} \ ,$$

on obtient un variété de fractions rationnelles difféomorphe par cette transformation à $\Omega_d \times \mathbb{T}^d$ avec

$$\Omega_d = \{(s_1, \dots, s_d) : \ s_1 > s_2 > \dots > s_d > 0\} \ ,$$

sur laquelle la forme symplectique s'exprime comme

$$\omega = \sum_{r=1}^{d} d\left(\frac{s_r^2}{2}\right) \wedge d\psi_r \ .$$

En utilisant cette transformation de Fourier non linéaire, Gérard et Grellier ont montré tout récemment le résultat de turbulence faible suivant[10, 17].

Théorème 0.2.3. *Il existe un sous–ensemble G_δ dense de données initiales dans $C_+^\infty(\mathbb{S}^1)$ tel que les solutions correspondantes de* (Szegő) *vérifient*

$$\forall s > \frac{1}{2} \ , \ \forall M > 0 \ , \ \limsup_{t \to \infty} \frac{\|u(t)\|_{H^s}}{|t|^M} = +\infty \ .$$

0.3 Une perturbation de l'équation de Szegő cubique

Dans la première partie de la thèse, on étudie l'équation de Szegő cubique perturbée par un potentiel linéaire,

$$\begin{cases} i\partial_t u = \Pi_+(|u|^2 u) + \alpha \displaystyle\int_{\mathbb{T}} u \ , \ x \in \mathbb{S}^1 \ , \ \alpha \in \mathbb{R} \\ u(0) = u_0 = \Pi(u_0) \ . \end{cases} \tag{α–Szegő}$$

L'équation (α–Szegő) est hamiltonienne d'énergie

$$E_\alpha(u) = \frac{1}{4}\|u\|_{L^4}^4 + \frac{\alpha}{2}|(u|1)|^2 \ .$$

L'équation(α–Szegő) est globalement bien posée dans $H_+^s(\mathbb{R})$ pour $s \geq 1/2$. Si $\alpha = 0$, l'équation est bien l'équation de Szegő cubique. L'un des avantages de choisir cette perturbation est qu'elle permet de garder l'une des deux paires de Lax.

Théorème 0.3.1. *L'équation* (α–Szegő) *admet la paire de Lax* (K_u, C_u). *Pour toute* $u \in C(\mathbb{R}, H^s_+)$, $s \geq \frac{1}{2}$, *solution de l'équation de* (α–Szegő), *on a l'identité*

$$\frac{d}{dt}K_u = [C_u, K_u] \ , \tag{0.3.1}$$

avec $C_u = \frac{i}{2}K_u^2 - iT_{|u|^2}$.

Comme dans le cas $\alpha = 0$, grâce à l'estimation de Peller dans $B^1_{1,1}$, la norme de L^∞ de la solution $u(t)$ est bornée par la norme de Sobolev H^s de l'initiale avec $s > 1$. En utilisant encore l'estimation de Gronwall, on obtient une borne L^∞ et une borne exponentielle pour les normes de Sobolev,

$$\|u(t)\|_{H^s} \leq C_s\|u_0\|_{H^s}\mathrm{e}^{C_s\|u_0\|^2_{H^s}|t|} \ , \ s > 1 \ .$$

Comme l'équation est non dispersive, la croissance exponentielle est moins étonnante. En fait, on va voir que cette croissance peut se produire.

Pour tout entier N, on définit des variété invariantes par

$$\mathcal{L}(N) := \left\{u \in H^{\frac{1}{2}}_+(\mathbb{S}^1) : \ \mathrm{rk}(K_u) = N\right\} \ . \tag{0.3.2}$$

Les éléments de $\mathcal{L}(N)$ sont des fractions rationnelles de la forme

$$u(z) = \frac{A(z)}{B(z)} \ , \ z \in \mathbb{S}^1 \ ,$$

où $A \in \mathbb{C}_N$, $B \in \mathbb{C}_N$ sont premiers entre eux, et les zéros de B sont tous de module > 1, et $\deg(A) = N$ ou $\deg(B) = N$.

Une conséquence de la paire de Lax est la conservation des valeurs propres σ_k^2 de K_u^2. En revanche, les valeurs propres de H_u^2 ne sont plus conservées, mais nous avons trouvé de nouvelles lois de conservation qui permettent de conclure à l'intégrabilité sur $\mathcal{L}(N)$.

Théorème 0.3.2. *Pour toute fonction* f *borélienne sur* $\mathbb{R}$,

$$L_f(u) := \Big(f(K_u^2)u|u\Big) - \alpha\Big(f(K_u^2)1|1\Big)$$

est une loi de conservation. Le flot de (α–Szegő) *est complètement intégrable sur* $\mathcal{L}(N)$ *pour tout* N.

En choisissant pour f la fonction indicatrice du singleton $\{\sigma_k^2\}$, on en déduit que

$$\ell_k(u) := \|u'_k\|^2 - \alpha\|v'_k\|^2$$

est conservée, où $\|\cdot\|$ est la norme de L^2, et u'_k, v'_k sont les projections de u et 1 sur l'espace $F_u(\sigma_k) := \ker(K_u^2 - \sigma_k^2\mathrm{I})$. Nos résultats concernent les symboles u avec H_u de rang fini, c'est–à–dire le flot sur $\mathcal{L}(N)$. Pour de tels u, rappelons que les ensembles

$$\Sigma_H(u) := \{\tau > 0 : \ u \not\perp E_u(\tau)\}, \quad \Sigma_K(u) := \{\tau \geq 0 : \ u \not\perp F_u(\tau)\} \ ,$$

avec $E_u(\tau) := \ker(H_u^2 - \tau^2 \mathrm{I})$, ont été définis au paragraphe précédent.

Un rôle particulier va être joué par la dynamique des valeurs propres ρ_j de H_u^2. Dans le cas $\alpha = 0$, les ρ_j sont conservées, mais pour $\alpha \neq 0$, les ρ_j vont varier au cours du temps, tout en restant chacune dans un intervalle délimité par deux valeurs propres σ_k consécutives. En particulier, pour un ensemble discret de temps, deux ρ_j peuvent se croiser, elles le font alors en prenant une même valeur σ_k, c'est–à–dire qu'il existe un temps t_0 tel que $\rho_1(t_0) = \rho_2(t_0) = \sigma_k \in \Sigma_H(u(t_0))$ avec $\rho_1(t) \neq \rho_2(t) \in \Sigma_H(u(t))$, $\sigma_k \in \Sigma_K(u(t))$, pour $0 < |t - t_0| << 1$. Lorsque $\alpha > 0$, on peut montrer qu'une condition nécessaire pour qu'un tel croisement ait lieu est $\ell_k < 0$.

Une partie importante de notre étude concerne le phénomène de turbulence faible. Les résultats obtenus sont résumés dans le théorème suivant.

Théorème 0.3.3. *Soit $u_0 \in \mathcal{L}(N)$, et soit u la solution de (α–Szegő) telle que $u(0) = u_0$. Une condition nécessaire pour que u soit turbulente est qu'il existe un entier $k \geq 1$, tel que $\ell_k(u_0) = 0$. En particulier, si $\alpha < 0$, l'équation (α–Szegő) n'admet pas de solution turbulente dans $\mathcal{L}(N)$, et les normes de Sobolev des solutions sont bornées uniformément en temps. En revanche, pour $\alpha > 0$, il existe des solutions u turbulentes dans $\mathcal{L}(N)$. En particulier, pour $u_0 \in \mathcal{L}(1)$, la solution u de l'équation de α–Szegő admet une croissance exponentielle en temps de la norme de Sobolev,*

$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{C_{\alpha,s}|t|},\ s > \frac{1}{2},\ C_{\alpha,s} > 0,\ |t| \to \infty \ , \tag{0.3.3}$$

si et seulement si

$$E_\alpha = \frac{1}{4}Q^2 + \frac{\alpha}{2}Q \ . \tag{0.3.4}$$

Pour les autres solutions sur $\mathcal{L}(1)$, les normes de Sobolev sont bornées.

Le théorème 0.3.3 est démontré dans les deux articles successifs [50, 48]. Le premier article est principalement consacré à la dynamique sur $\mathcal{L}(1)$, et le deuxième établit le théorème 0.3.2 ainsi que ce qui concerne la dynamique sur $\mathcal{L}(N)$ pour $N \geq 2$ dans le théorème 0.3.3.

Il est à noter que le résultat de croissance des normes de Sobolev par Gérard et Grellier est exprimé en terme de limite supérieure ; en d'autres termes, il existe des possibilités d'intermittence pour les oscillations en temps. En revanche, nos résultats décrivent la convergence vers l'infini des normes en temps. Il existe un autre exemple de turbulence faible de ce type, obtenu par Oana Pocovnicu [44], qui a étudié l'équation de Szegő cubique sur $\mathbb{R}$, mais la croissance connue est seulement polynomiale en t^{2s-1}. Notre résultat est le premier exemple de croissance exponentielle en temps pour de telles équations.

0.4 Une équation mixte Schrödinger–ondes

La deuxième partie est consacrée au système hamiltonien

$$\begin{cases} (i\partial_t + \mathcal{A})\, U = |U|^2 U \ , \ (x,y) \in \mathbb{R} \times \mathbb{T} \ , \\ U(0,x,y) = U_0(x,y) \ , \ U_0(x,y) = -U_0(x, y+\pi) \ , \end{cases} \tag{WS}$$

où $\mathcal{A} = \partial_{xx} - |D_y|,\ |D_y| = \sqrt{-\partial_{yy}}$.

Le hamiltonien de l'équation est

$$H(U) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} (|\partial_x U(x,y)|^2 + |D_y|U(x,y)\overline{U}(x,y))\,dx\,dy + \frac{1}{4}\int_{\mathbb{R}\times\mathbb{T}} |U(x,y)|^4\,dx\,dy\ .$$

En appliquant la stratégie de Hani–Pausader–Tzvetkov–Visciglia [26], on montre que la dynamique asymptotique des petites solutions de (WS) est liée aux solutions du système résonant ci–dessous,

$$\begin{cases} i\partial_t G_\pm(t) = \mathcal{R}[G_\pm(t), G_\pm(t), G_\pm(t)]\ , \\ \mathcal{F}_\mathbb{R}\mathcal{R}[G_\pm, G_\pm, G_\pm](\xi, y) = \Pi_\pm(|\widehat{G}_\pm|^2\widehat{G}_\pm)(\xi, y)\ , \end{cases} \tag{0.4.1}$$

où $\widehat{G}(\xi,\cdot) = \mathcal{F}_\mathbb{R} G(\xi,\cdot)$.

Théorème 0.4.1. *Pour un entier N assez grand, on considère les espaces de Banach S et S^+ de fonctions sur $\mathbb{R}\times\mathbb{T}$,*

$$\|F\|_S := \|F\|_{H^N_{x,y}} + \|xF\|_{L^2_{x,y}}\ , \quad \|F\|_{S^+} := \|F\|_S + \|(1-\partial_{xx})^4 F\|_S + \|xF\|_S\ . \tag{0.4.2}$$

Il existe $\varepsilon > 0$ telle que si $U_0 \in S^+$ satisfait à

$$\|U_0\|_{S^+} \le \varepsilon\ , \tag{0.4.3}$$

(1) *Si $\widetilde{G}$ est une solution de (0.4.1) avec la donnée initiale U_0, alors, il existe une solution unique U de (WS) telle que $\mathrm{e}^{-it\mathcal{A}}U(t) \in C([0,\infty), S)$ et*

$$\|\mathrm{e}^{-it\mathcal{A}}U(t) - \widetilde{G}(\pi \ln t)\|_S \to 0 \ si\ \ t \to +\infty\ .$$

(2) *Réciproquement, si on considère la solution correspondante U de (WS) avec la donnée initiale U_0 satisfaisant à (0.4.3), et ε assez petit, alors il existe une solution $\widetilde{G}$ de (0.4.1), telle que*

$$\|\mathrm{e}^{-it\mathcal{A}}U(t) - \widetilde{G}(\pi \ln t)\|_S \to 0 \ si\ t \to +\infty\ . \tag{0.4.4}$$

La différence essentielle entre le modèle de [26] et notre modèle est que celui–ci est non dispersif, et la perte d'estimation de Strichartz rend la stratégie de [26] plus délicate à appliquer. Le point clé qui nous permet de mener à bien ce programme est l'utilisation de l'estimation de Peller dans $B^1_{1,1}$.

Comme conséquence du théorème 0.4.1 et du résultat [17] de Gérard–Grellier, on obtient le résultat de turbulence faible suivant.

Théorème 0.4.2. *Il existe des solutions $U(t)$ de (WS), avec une petite donnée initiale dans S^+, telles que*

$$\limsup_{t\to\infty} \frac{\|U(t)\|_{L^2_x H^s_y}}{(\log|t|)^M} = \infty\ , \forall s > \frac{1}{2}\ ,\ \forall M \in \mathbb{N}\ .$$

Les démonstrations font l'objet du troisième article [49]. Remarquons que, compte tenu des estimées de [11] déjà rappelées, les normes Sobolev de ces solutions sont en général, pour des données petites dans S^+, en $O(t^\delta)$ avec δ petit. On obtient donc une croissance presque optimale.

0.5 Perspectives et problèmes ouverts

La principal but de l'étude de l'équation (α–Szegő) est la dynamique de la solution générale. Dans ce contexte, on a observé le phénomène de la turbulence faible de certaines solutions rationnelles. Un important problème ouvert est d'obtenir de nouvelles informations sur les solutions de rang infini. En particulier, nous sommes intéressés par l'existence de données génériques turbulentes avec une convergence des normes Sobolev vers l'infini : est-ce qu'on a un théorème comme celui de Gérard et Grellier [17], avec cette fois une vraie convergence des normes Sobolev vers l'infini ?

Une autre question ouverte concerne l'équation de Szegő cubique perturbée par d'autres fonctions de la moyenne de u, c'est–à–dire un hamiltonien de la forme

$$E(u) = \frac{1}{4}\|u\|_{L^4}^4 + \frac{1}{2}F(|(u|1)|^2) \ ,$$

avec une fonction F non linéaire. Dans ce cadre, la paire de Lax pour K_u reste vraie, mais les lois de conservation L_f disparaissent ! La question qui se pose est de savoir si le système est encore intégrable sur $\mathcal{L}(N)$. Il serait également utile de connaître à quelles conditions sur F il existe des solutions turbulentes.

En ce qui concerne l'équation mixte Schrödinger–ondes, la question la plus brûlante est de savoir si elle admet des solutions dont les normes de Sobolev croissent vers l'infini sur $\mathbb{R} \times \mathbb{R}$. Comme nous l'avons déjà rappelé, de tels résultats ont été obtenus par Oana Pocovnicu dans sa thèse [43, 44], pour l'équation de Szegő cubique sur la droite. Or l'équation de demi–onde sur $\mathbb{R}$ est approchée par l'équation de Szegő sur $\mathbb{R}$ [46]. Il serait donc utile d'établir une approximation de la solution de l'équation mixte Schrödinger–ondes pour un temps plus long, voire infini, sur le plan $\mathbb{R}_x \times \mathbb{R}_y$, pour des données assez petites.

Enfin, pour les solutions de l'équation de demi–onde sur $\mathbb{T}$, on dispose déjà d'une croissance de la norme H^s, $s > \frac{1}{2}$, depuis ε jusqu'à $\varepsilon|\log \varepsilon|^{2s-1}$ [12] ; mais on ne sait pas si cette norme peut croître jusqu'à l'infini le long d'une trajectoire. Le problème de solutions turbulentes pour cette équation est encore ouvert.

Chapitre 1

Large time blow up for a perturbation of the cubic Szegő equation

Ce chapitre est la reprise d'un article à paraître dans le journal "Analysis and PDEs".

1.1 Introduction

The study on the long time behavior of solutions of Schrödinger type Hamiltonian equations is a central issue in the theory of dispersive nonlinear partial differential equations. For instance, Colliander, Keel, Staffilani, Takaoka and Tao studied the following cubic defocusing nonlinear Schrödinger equation in [7],

$$i\partial_t u + \Delta u = \pm |u|^2 u \ , \quad (t,x) \in \mathbb{R} \times \mathbb{T}^2 \ . \tag{1.1.1}$$

In that paper, they constructed solutions with small H^s norm at the initial moment, which present a large Sobolev H^s norm at a sufficiently long time T. Guardia and Kaloshin improved this result by refining the estimates on the time T [24]. Zaher Hani studied a version of nonlinear Schrödinger equation obtained by canceling the least resonant part, and showed the existence of unbounded trajectories in high Sobolev norms [25]. Recently, Hani, Pausader, Tzvetkov and Visciglia studied the nonlinear Schrödinger equation (1.1.1) on the spatial domain $\mathbb{R} \times \mathbb{T}^d$, and obtained global solutions to the defocusing and focusing problems on (for any $d \geq 2$) with infinitely growing high Sobolev norms H^s [26].

There is another related result by Gérard and Grellier [12]. They considered the following degenerate half wave equation on the one dimensional torus,

$$i\partial_t u - |D| u = |u|^2 u \ . \tag{1.1.2}$$

They found solutions with small Sobolev norms at initial time which become much larger as time grows. More precisely, there exist sequences of solutions u^n and $t^n \to \infty$ such that $\|u_0^n\|_{H^r} \to 0$ for any r, but

$$\|u^n(t^n)\|_{H^s} \sim \|u_0^n\|_{H^s} \left(\log \frac{1}{\|u_0^n\|_{H^s}} \right)^{2s-1} , \ s > 1 \ .$$

In fact, the above result is a consequence of the studies on the so-called *cubic Szegő equation* which is introduced by Gérard and Grellier as a model of non-dispersive dynamics [11, 13],

$$i\partial_t u = \Pi(|u|^2 u) \ . \tag{1.1.3}$$

The above equation turns out to be the resonant part of the half wave equation (1.1.2). The operator Π, which is the so-called Szegő operator, is defined as a projector onto the non-negative frequencies. If $u \in \mathcal{D}'(\mathbb{S}^1)$ is a distribution on the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, then

$$\Pi(u) = \Pi\big(\sum_{k\in\mathbb{Z}} \widehat{u}(k)\mathrm{e}^{ik\theta}\big) = \sum_{k\geq 0} \widehat{u}(k)\mathrm{e}^{ik\theta}. \tag{1.1.4}$$

Notice that, on the Hilbert space $L^2(\mathbb{S}^1)$ endowed with the inner product

$$(u \mid v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\mathrm{e}^{ix})\overline{v(\mathrm{e}^{ix})}dx \ , \tag{1.1.5}$$

Π is the orthogonal projector on the subspace $L^2_+(\mathbb{S}^1)$ defined by the conditions

$$\forall k < 0,\ \widehat{u}(k) = 0 \ .$$

Gérard and Grellier studied the Szegő equation on the space $H^{\frac{1}{2}}(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1) := H^{\frac{1}{2}}_+(\mathbb{S}^1)$ and displayed two Lax pair structures for this completely integrable system [11, 13]. Moreover, they established an explicit formula of every solution with rational initial data [16] and illustrated the large time behavior of Sobolev norms of the solutions, for instance,

Theorem 1.1.1. *[11] Every solution u of (1.1.3) on*

$$\widetilde{\mathcal{M}}(1) := \big\{u = \frac{a+bz}{1-pz} :\ 0 \neq a \in \mathbb{C}, b \in \mathbb{C}, p \in \mathbb{C}, |p| < 1, a + bp \neq 0\big\}$$

satisfies

$$\forall s > \frac{1}{2},\ \sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} < \infty.$$

However, there exists a family of Cauchy data u_0^ε in $\widetilde{\mathcal{M}}(1)$ which converges in $\widetilde{\mathcal{M}}(1)$ for the $C^\infty(\mathbb{S}^1)$ topology as $\varepsilon \to 0$, and $K > 0$ such that the corresponding solutions of (1.1.3) u^ε satisfy

$$\forall \varepsilon > 0,\ \exists t^\varepsilon > 0,\ \|u^\varepsilon(t^\varepsilon)\|_{H^s} \geq K(t^\varepsilon)^{2s-1} \ \text{as}\ t^\varepsilon \to \infty,\ \forall s > \frac{1}{2}.$$

Another result on this Szegő equation was obtained by Pocovnicu [45, 44], who studied this equation by replacing the circle $\mathbb{S}^1$ with the real line and got a polynomial growth of high Sobolev norms (Corollary 4, [44]), which says that there exists a solution u of the Szegő equation and a constant $C > 0$ such that $\|u(t)\|_{H^s} \geq C|t|^{2s-1}$ for sufficiently large $|t|$.

The aim of this chapter is to study the properties of global solutions for the following Hamiltonian equation on $L^2_+(\mathbb{S}^1)$, which is the cubic Szegő equation with a linear perturbation,

$$\begin{cases} i\partial_t u = \Pi(|u|^2u) + \alpha(u|1), \ \alpha \in \mathbb{R} , \\ u(0,x) = u_0(x) . \end{cases} \tag{1.1.6}$$

Recall that, in view of the above definition (1.1.5),

$$(u|1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\mathrm{e}^{i\theta})d\theta$$

is the average of u on $\mathbb{S}^1$.

The equation (1.1.6), called the α–Szegő equation, inherits three formal conservation laws :

$$\begin{aligned} &\text{mass} : Q(u) := \int_{\mathbb{S}^1} |u|^2 \frac{d\theta}{2\pi} = \|u\|^2_{L^2} , \\ &\text{momentum} : M(u) := (Du|u), \ D := -i\partial_\theta = z\partial_z , \\ &\text{energy} : E_\alpha(u) := \frac{1}{4}\int_{\mathbb{S}^1} |u|^4 \frac{d\theta}{2\pi} + \frac{1}{2}\alpha|(u|1)|^2 . \end{aligned}$$

Slight modifications of the proof of the well-posedness result in [11] lead to the result that the α–Szegő equation is globally well-posed in $H^s_+(\mathbb{S}^1) = H^s(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$ for $s \geq \frac{1}{2}$ as follows :

Theorem 1.1.2. *Given $u_0 \in H^{\frac{1}{2}}_+(\mathbb{S}^1)$, there exists a unique global solution $u \in C(\mathbb{R}; H^{\frac{1}{2}}_+)$ of (1.1.6) with u_0 as the initial condition. Moreover, if $u_0 \in H^s_+(\mathbb{S}^1)$ for some $s > \frac{1}{2}$, then $u \in C^\infty(\mathbb{R}; H^s_+)$. Furthermore, if $u_0 \in H^s_+(\mathbb{S}^1)$ with $s > 1$, the Wiener norm of u is bounded uniformly in time,*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_W := \sup_{t\in\mathbb{R}} \sum_{k=0}^{\infty} |\widehat{u(t)}(k)| \leq C_s \|u_0\|_{H^s} . \tag{1.1.7}$$

Now, we present our main results. In our case with a perturbation term, we gain the following statement that for the case $\alpha < 0$ the Sobolev norm stays bounded uniformly in time, while for $\alpha > 0$, it may grow exponentially fast.

Theorem 1.1.3. *Let $u_0 = b_0 + \frac{c_0 z}{1-p_0 z}$, $c_0 \neq 0$, $|p_0| < 1$.*

For $\alpha < 0$, the Sobolev norm of the solution will stay bounded,

$$\|u(t)\|_{H^s} \leq C, \ C \text{ does not depend on time } t, \ s \geq 0 . \tag{1.1.8}$$

For $\alpha > 0$, the solution u of the α–Szegő equation (1.1.6) has a Sobolev norm growing exponentially in time,

$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{C_{\alpha,s}|t|}, \ s > \frac{1}{2}, \ C_{\alpha,s} > 0, \ |t| \to \infty , \tag{1.1.9}$$

if and only if

$$E_\alpha = \frac{1}{4}Q^2 + \frac{\alpha}{2}Q . \tag{1.1.10}$$

Remark 1.1.1. *Here are several remarks :*

1. *Together with the results in [11, 13], we now have a complete picture for the high Sobolev norm of the solutions to the α–Szegő equation. For $\alpha < 0$, it stays bounded (uniformly on time), for $\alpha > 0$, it turns out to have an exponential growth for some initial data satisfying the condition in the Theorem 1.1.3. Finally, for $\alpha = 0$, the trajectories of the Szegő equation with rational initial data are quasiperiodic with instability of the H^s norm as in Theorem 1.1.1.*
2. *Our result is in strong contrast with Bourgain's and Staffilani's results for the dispersive equations in [3, 47], which say that the dispersive equations admit polynomial upper bounds on Sobolev norms. Here, we give an example of exponential growth of Sobolev norms for a non dispersive model.*
3. *The solutions to the α–Szegő equation admit an exponential upper bound of the Sobolev norms. Assume $s > 1$, it is easy to solve (1.1.6) locally in time. More precisely, one has to solve the integral equation*

$$u(t) = u_0 - i\int_0^t (\Pi(|u|^2u) + \alpha(u|1))dt' \ .$$

Thus

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + c\int_0^t (1 + \|u(t')\|_W^2)\|u(t')\|_{H^s}dt' \ ,$$

since by Theorem 1.1.2, the Wiener norm is uniformly bounded, then by Gronwall's inequality, we have

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s}\mathrm{e}^{ct} \ .$$

This shows that estimate (1.1.9) is the worst that can happen.

This chapter is organized as follows. In section 2, we prove that there exists a Lax pair for the α–Szegő equation based on Hankel operators. Then we define the manifolds $\mathcal{L}(k) := \{u : \operatorname{rk}(K_u) = k,\ k \in \mathbb{Z}^+\}$ with the shifted Hankel operator K_u. These manifolds are proved to be invariant by the flow and can be represented as sets of rational functions. In this paper we will just consider the solutions $u \in \mathcal{L}(1)$. We plan to address the other cases in a forthcoming work. In section 3, we prove the large time blow up result and the boundedness of the Wiener norm to show that our result is optimal. Furthermore, we provide an example which describes the energy cascade. Finally, we present some perspectives in section 4.

1.2 The Lax pair structure

To introduce the Lax pair structure, let us first define some useful operators and notation. For $X \subset \mathcal{D}'(\mathbb{S}^1)$, we denote

$$X_+(\mathbb{S}^1) := \left\{ u(\mathrm{e}^{i\theta}) \in X,\ u(\mathrm{e}^{i\theta}) = \sum_{k\geq 0} \widehat{u}(n)\mathrm{e}^{ik\theta} \right\} . \qquad (1.2.1)$$

For example, L^2_+ denotes the Hardy space of L^2 functions which extend to the unit disc $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$ as holomorphic functions

$$u(z) = \sum_{k\geq 0} \widehat{u}(n) z^k,\ \sum_{k\geq 0} |\widehat{u}(n)|^2 < \infty\ . \tag{1.2.2}$$

An element of L^2_+ can therefore be seen either as a square integrable function $u = u(\mathrm{e}^{i\theta})$ on the circle with only non negative Fourier modes, or a holomorphic function $u = u(z)$ on the unit disc with square summable Taylor coefficients. The Szegő operator Π defined as (1.1.4) is an orthogonal projector $L^2(\mathbb{S}^1) \to L^2_+(\mathbb{S}^1)$.

Using the Szegő projector , we first introduce two important classes of operators on $L^2_+(\mathbb{S}^1)$, namely, the Hankel and Toeplitz operators. One may refer to [11, section 3] for more details on these operators, or see [38, 41] for general references.

By a Hankel operator we mean a bounded operator Γ on the sequence space ℓ^2 which has a Hankel matrix in the standard basis $\{e_j\}_{j\geq 0}$,

$$(\Gamma e_j,\ e_k\ ) = \gamma_{j+k},\ j,\ k \geq 0\ , \tag{1.2.3}$$

where $\{\gamma_j\}_{j\geq 0}$ is a sequence of complex numbers. Let S be the shift operator on ℓ^2,

$$Se_j = e_{j+1},\ j \geq 0\ .$$

It is easy to show that a bounded operator Γ on ℓ^2 is a Hankel operator if and only if

$$S^*\Gamma = \Gamma S\ . \tag{1.2.4}$$

Definition 1.2.1. *For any given $u \in H^{\frac{1}{2}}_+(\mathbb{S}^1)$, $b \in L^\infty(\mathbb{S}^1)$, we define two operators $H_u,\ T_b :\ L^2_+ \to L^2_+$ as follows. For any $h \in L^2_+$,*

$$H_u(h) = \Pi(u\bar{h})\ , \tag{1.2.5}$$
$$T_b(h) = \Pi(bh)\ . \tag{1.2.6}$$

Notice that H_u is $\mathbb{C}$–anti-linear and symmetric with respect to the real scalar product $\mathrm{Re}(u|v)$. In fact, it satisfies

$$(H_u(h_1)|h_2) = (H_u(h_2)|h_1)\ ,$$

T_b is $\mathbb{C}$–linear and is self-adjoint if and only if b is real-valued.

It is easy to show that H_u is a Hankel operator while T_b is a Toeplitz operator. Indeed, H_u is given in terms of Fourier coefficients by

$$\widehat{H_u(h)}(k) = \sum_{\ell\geq 0} \widehat{u}(k+\ell)\overline{\widehat{h}(\ell)}\ , \tag{1.2.7}$$

then

$$\begin{aligned} S^*H_u(h) &= \sum_{k,\ell\geq 0} \widehat{u}(k+\ell)\overline{\widehat{h}(\ell)}S^*e_k = \sum_{k,\ell\geq 0} \widehat{u}(k+\ell+1)\overline{\widehat{h}(\ell)}e_k\ , \\ H_uSh &= \sum_{k\geq\ell,\ell\geq 0} \widehat{u}(k)e_k\overline{\widehat{h}(\ell)e_{\ell+1}} = \sum_{k,\ell\geq 0} \widehat{u}(k+\ell+1)\overline{\widehat{h}(\ell)}e_k\ , \end{aligned}$$

then $S^*H_u = H_u S$, thus H_u is a Hankel operator. We may also represent T_b in terms of Fourier coefficients,

$$\widehat{T_b(h)}(k) = \sum_{\ell \geq 0} \widehat{b}(k-\ell)\widehat{h}(\ell) \ ,$$

then its matrix representation, in the basis $e_k, k \geq 0$, has constant diagonals, T_b is a Toeplitz operator.

Moreover, by (1.2.7), we have

$$\widehat{H_u^2(h)}(n) = \sum_{\ell \geq 0} c_{n\ell}\widehat{h}(\ell) \ , \ c_{n\ell} = \sum_{p \geq 0} \widehat{u}(n+p)\overline{\widehat{u}(p+\ell)} \ .$$

Hence,

$$\mathrm{Tr}(H_u^2) = \sum_{n \geq 0} c_{nn} = \sum_{n \geq 0} (1+|n|)|\widehat{u}(n)|^2 = Q(u) + M(u) \ .$$

Thus for $u \in H_+^{\frac{1}{2}}$, H_u is a Hilbert-Schmidt operator with

$$\mathrm{Tr}(H_u^2) = \sum_{n=0}^{\infty} (n+1)|\widehat{u}(n)|^2 \ . \tag{1.2.8}$$

We now define another operator $K_u := T_z^* H_u$. In fact T_z is exactly the shift operator S as above, we then call it the shifted Hankel operator, which satisfying the following identity

$$K_u^2 = H_u^2 - (\cdot \mid u)u \ . \tag{1.2.9}$$

By using these operators, we are able to construct the Lax pair structure for our model equations. Firstly, the cubic Szegő equation was proved to admit two Lax pairs[11, 13].

Theorem 1.2.1 ([11], Theorem 3)**.** *Let $u \in C(\mathbb{R}, H^s(\mathbb{S}^1))$ for some $s > \frac{1}{2}$. The cubic Szegő equation*

$$i\partial_t u = \Pi(|u|^2 u) \tag{1.2.10}$$

has two Lax pairs (H_u, B_u) and (K_u, C_u), namely, if u solves (1.2.10), then

$$\frac{dH_u}{dt} = [B_u, H_u] \ , \ \frac{dK_u}{dt} = [C_u, K_u] \ , \tag{1.2.11}$$

where

$$B_u = \frac{i}{2}H_u^2 - iT_{|u|^2} \ , \ C_u = \frac{i}{2}K_u^2 - iT_{|u|^2} \ .$$

When $\alpha \neq 0$, the new model equation inherit one of the Lax pairs while the other one is ruined.

Corollary 1.2.1. *The perturbed Szegő equation (1.1.6) with $\alpha \neq 0$ still has one Lax pair (K_u, C_u).*

Démonstration. The proof is based on the following identity ([16], Lemma 1),

$$H_{\Pi(|u|^2u)} = T_{|u|^2}H_u + H_uT_{|u|^2} - H_u^3\ . \tag{1.2.12}$$

Using equation (1.1.6) and (1.2.12),

$$\frac{dH_u}{dt} = H_{-i\Pi(|u|^2u)-i\alpha(u|1)} = -i(T_{|u|^2}H_u + H_uT_{|u|^2} - H_u^3) - i\alpha(u|1)H_1\ .$$

Using the anti-linearity of H_u, we deduce that

$$\frac{dH_u}{dt} = [B_u, H_u] - i\alpha(u|1)H_1\ , \tag{1.2.13}$$

which means that (H_u, B_u) is no longer a Lax pair. Fortunately, we have $T_z^*H_1 = 0$, which leads to the following identity

$$\frac{dK_u}{dt} = [C_u, K_u]\ . \tag{1.2.14}$$

□

An important consequence of this Lax pair structure is the existence of finite dimensional submanifolds of $L^2_+(\mathbb{S}^1)$ which are invariant by the flow of (1.1.6). To describe these manifolds, Gérard and Grellier (Appendix 4, [11]) proved a Kronecker-type theorem that, the Hankel operator H_u is of finite rank k if and only if u is a rational function of the complex variable z, with no poles in the unit disc, and of the form $u(z) = \frac{A(z)}{B(z)}$ with $A \in \mathbb{C}_{k-1}[z]$, $B \in \mathbb{C}_k[z]$, $B(0) = 1$, $\deg(A) = k - 1$ or $\deg(B) = k$, A and B have no common factors and $B(z) \neq 0$ if $|z| \leq 1$. In fact, we can prove a similar theorem for our case.

Definition 1.2.2. *Let k be a positive integer, we define*

$$\mathcal{L}(k) := \left\{u \in H_+^{\frac{1}{2}}(\mathbb{S}^1) :\ \mathrm{rk}(K_u) = k\right\}\ . \tag{1.2.15}$$

Due to the Lax pair structure, the manifolds $\mathcal{L}(k)$ are invariant by the flow.

Theorem 1.2.2. *$u \in \mathcal{L}(k)$ if and only if u is a rational function satisfying*

$$u(z) = \frac{A(z)}{B(z)}\ \text{with}\ A, B \in \mathbb{C}_k[z], A\wedge B = 1, \deg(A) = k\ \text{or}\ \deg(B) = k, B^{-1}(\{0\})\cap\overline{D} = \emptyset\ ,$$

where $A \wedge B = 1$ means A and B have no common factors.

Démonstration. The proof is based on the results by Gérard and Grellier (see Appendix 4, [11]), they proved that

$$\begin{aligned}\mathcal{M}(k+1) &= \{u :\ \mathrm{rk}(H_u) = k+1\}\\ &= \left\{\begin{array}{l} u(z) = \dfrac{A(z)}{B(z)} :\ A \in \mathbb{C}_k[z], B \in \mathbb{C}_{k+1}[z], B(0) = 1, \deg(A) = k \\ \text{or}\ \deg(B) = k+1, A \wedge B = 1,\ B^{-1}(0) \cap \overline{D} = \emptyset \end{array}\right\}.\end{aligned}$$

For $u \in \mathcal{M}(k+1)$, $\dim \mathrm{Im} H_u = k+1$, then $u, T_z^* u, \cdots, (T_z^*)^{k+1} u$ are linearly dependent, i.e, there exist C_ℓ, not all zero, such that $\sum_{\ell=0}^{k+1} C_\ell (T_z^*)^\ell u = 0$. We get

$$\sum_{\ell=0}^{k+1} C_\ell \widehat{u}(\ell+n) = 0 \ , \ \forall n \geq 0 \ .$$

This is a recurrent equation for the sequence $\widehat{u}$. It can be solved by means of elementary linear algebra. Define

$$P(X) = \sum_{\ell=0}^{k+1} C_\ell X^\ell = C \prod_{p \in \mathcal{P}} (X-p)^{m_p} \ ,$$

where $\mathcal{P} = \{p \in \mathbb{C} : P(p) = 0\}$ and m_p is the multiplicity of p.
$(\widehat{u}(n))_{n \geq 0}$ is a linear combination of the following sequences :

$$n^\ell p^{n-\ell}, \ p \neq 0, \ 0 \leq \ell \leq m_p - 1 \ ,$$
$$\delta_{nm}, \ p = 0, \ 0 \leq m \leq m_0 - 1 \ .$$

Recall that

$$u(z) = \sum_{n \geq 0} \widehat{u}(n) z^n \text{ for } |z| < 1 \ ,$$

then u is a linear combination of $\frac{1}{(1-pz)^{\ell+1}}$ with $0 < |p| < 1$ for $0 \leq \ell \leq m_p - 1$, and of z^ℓ for $0 \leq \ell \leq m_0 - 1$.
Consequently, $u(z) = \frac{A(z)}{B(z)}$ with

$$\begin{aligned} &\deg(A) \leq k, \ \deg(B) = k+1, \ \text{if } p \neq 0, \ p \in \mathcal{P} \ , \\ &\deg(A) = k, \ \deg(B) \leq k, \ \text{if } 0 \in \mathcal{P} \ . \end{aligned}$$

Note that

$$0 \in \mathcal{P}$$

is equivalent to

$$1 \in \mathrm{Im} H_u$$

or to

$$\ker K_u \cap \mathrm{Im} H_u \neq \{0\} \ ,$$

since $K_u = T_z^* H_u$, $\mathrm{rk}(H_u) - 1 \leq \mathrm{rk}(K_u) \leq \mathrm{rk}(H_u)$. For $u \in \mathcal{L}(k)$, $\mathrm{rk}(K_u) = k$, then $u = \frac{A(z)}{B(z)}$ with

$$\begin{aligned} &\deg(A) \leq k-1, \deg(B) = k, \ \text{if } \mathrm{rk}(H_u) = \mathrm{rk}(K_u) = k \ , \\ &\deg(A) = k, \deg(B) \leq k, \ \text{if } \mathrm{rk}(H_u) = \mathrm{rk}(K_u) + 1 = k+1 \ . \end{aligned}$$

The proof of the converse is similar. So

$$\begin{aligned} \mathcal{L}(k) &= \{u : \ \mathrm{rk}(K_u) = k+1\} \\ &= \left\{ \begin{array}{l} u(z) = \dfrac{A(z)}{B(z)} : \ A \in \mathbb{C}_k[z], B \in \mathbb{C}_k[z], B(0) = 1, \deg(A) = k \\ \text{ or } \ \deg(B) = k, A \wedge B = 1, \ B^{-1}(0) \cap \overline{D} = \emptyset \end{array} \right\} . \end{aligned}$$

The proof is completed. $\square$

1.3 The proof of the main theorem

In this section, we will prove that the α–Szegő equation (1.1.6) admits the large time blow up as in Theorem 1.1.3, we will also give an example to describe this phenomenon in terms of energy transfer to high frequencies. Before the proof of the main theorem, let us prove the boundedness of Wiener norm as in Theorem 1.1.2.

Let us recall the definition of $B^s_{p,q}(\mathbb{S}^1)$. Let $\chi \in C^\infty_c(\mathbb{R}^+)$ satisfy

$$\begin{cases} 0 \le \chi \le 1 \ , \\ \chi(t) = 1 \ , \ t < 1 \ , \\ \chi(t) = 0 \ , \ t > 2 \ . \end{cases}$$

Set ψ_j's as

$$\psi_0(t) = 1 - \chi(t) \ , \ \psi_j(t) = \chi(2^{-j+1}t) - \chi(2^{-j}t) \ .$$

Define the operator Δ_j for $f \in \mathcal{D}'(\mathbb{S}^1)$ as

$$\Delta_j f = \sum_{k\in\mathbb{Z}} \psi_j(k)\widehat{f}(k)\,\mathrm{e}^{ik\theta} \ .$$

Then the Besov space is defined as

$$B^s_{p,q}(\mathbb{S}^1) := \left\{ f \in \mathcal{D}'(\mathbb{S}^1) : 2^{js}\|\Delta_j f\|_{L^p} \in \ell^q_j, 1 \le p, q \le +\infty,\ 0 \le j \le +\infty \right\} ,$$

with the norm $\|f\|_{B^s_{p,q}(\mathbb{S}^1)} = \big(\sum\limits_{j=0}^{+\infty}(2^{js}\|\Delta_j f\|_{L^p})^q\big)^{\frac{1}{q}}$.

Proposition 1.3.1. *Assume $u_0 \in H^s_+(\mathbb{S}^1)$ with $s > 1$, let u be the corresponding unique solution of (1.1.6). Then*

$$\|u(t)\|_W \le C_s\|u_0\|_{H^s}, \ \forall t \in \mathbb{R} \ .$$

Démonstration. By Peller's theorem [41], the regularity of u ensures that H_u is trace class and the trace norm of H_u is equivalent to the $B^1_{1,1}$ norm of u. It is easy to show that the $B^1_{1,1}(\mathbb{S}^1)$ norm can be controlled by the H^s norm of u with $s > 1$. Indeed, there exist C, $C_s\ > 0$, such that

$$\begin{aligned} \|u\|_{B^1_{1,1}} &= \sum_{j=0}^{+\infty} 2^j\|\Delta u\|_{L^1} \le C\sum_{j=0}^{+\infty} 2^j\|\Delta u\|_{L^2} \\ &\le C\Big(\sum_{j=0}^{+\infty} 2^{2js}\|\Delta u\|^2_{L^2}\Big)^{\frac{1}{2}}\Big(\sum_{j=0}^{+\infty} 2^{2j(1-s)}\Big)^{\frac{1}{2}} \\ &\le C_s\|u\|_{H^s}, \ \forall s > 1 \ . \end{aligned} \tag{1.3.1}$$

So if $u \in H^s$ with $s > 1$, then H_u is trace class with

$$\mathrm{Tr}(|H_u|) \le C_s\|u\|_{H^s} \ .$$

Since $K_u = T_z^* H_u$, then
$$K_u^2 = H_u^2 - (\cdot \mid u)u \ ,$$
then
$$\mathrm{Tr}(|K_u|) \leq \mathrm{Tr}(|H_u|) \ .$$
Due to the Lax pair structure, we get $K_{u(t)}$ is isospectral to K_{u_0}, then
$$\mathrm{Tr}(|K_{u(t)}|) = \mathrm{Tr}(|K_{u_0}|) \ ,$$
so
$$\mathrm{Tr}(|K_{u(t)}|) \leq C_s \|u_0\|_{H^s} \ .$$
Since $\|u\|_W = |\widehat{u}(0)| + \sum_{n\geq 1} |\widehat{u}(n)|$ and $|\widehat{u}(0)| \leq \|u\|_{L^2}$, we just need to show that
$$\sum_{n\geq 1} |\widehat{u}(n)| \leq C\mathrm{Tr}(|K_u|) \ .$$

Let e_n as the orthonormal basis of L^2_+, then for any bounded operator B,
$$\sum_n \big|(K_u e_n | B e_n)\big| \leq \mathrm{Tr}(|K_u|)\|B\|.$$
Then we gain that $\sum_{n\geq 1} |\widehat{u}(2n)| + \sum_{n\geq 1} |\widehat{u}(2n+1)| \leq \mathrm{Tr}(|K_u|)$, by taking $B = T_z$ and $B = Id$. This completes the proof. $\square$

Remark 1.3.1. *In fact, to prove the global wellposedness, it is natural to use the Brezis-Gallouët type estimate (Appendix 2, [11]), for $s > \frac{1}{2}$*
$$\|u\|_W \leq C_s \|u\|_{H^{\frac{1}{2}}} \big[\log(1 + \frac{\|u\|_{H^s}}{\|u\|_{H^{1/2}}})\big]^{\frac{1}{2}} \ ,$$
which leads to a double exponential on time growth for the Sobolev norm of u. Fortunately, by the estimate in Proposition 1.3.1, we know the H^s norm of the solutions will admit an exponential on time upper bound for $s > 1$ (see Remark 1.1.1).

Now, let us start the large time blow up theorem.

Theorem 1.3.1. *For $\alpha > 0$, we consider the solution of the Szegő equation (1.1.6) with initial data $u_0 \in \mathcal{L}(1)$.*

1. *If the trajectory issued from u_0 is not relatively compact in $\mathcal{L}(1)$, then*
$$E_\alpha = \frac{1}{4}Q^2 + \frac{\alpha}{2}Q \ . \tag{1.3.2}$$

2. *If (1.3.2) holds, then*
$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{C_{\alpha,s}|t|}, \ s > \frac{1}{2}, \ C_{\alpha,s} > 0, \ |t| \to \infty \ . \tag{1.3.3}$$

Remark 1.3.2. *From the theorem, the equality (1.3.2), which is invariant by the flow, is a necessary and sufficient condition to cause the large time blow up.*

Démonstration. First, since the trajectory of the solution is not relatively compact in $\mathcal{L}(1)$, the level set $L(u_0) := \{u \in \mathcal{L}(1) : \ Q(u) = Q(u_0), \ M(u) = M(u_0), \ E_\alpha(u) = E_\alpha(u_0)\}$ is not compact in $\mathcal{L}(1)$.

We rewrite $u \in \mathcal{L}(1)$ as

$$u = b + \frac{cz}{1 - pz} \ ,$$

then the conservation laws under the coordinates $b, \ p, \ c$ are given as

$$\begin{aligned} Q &= \|u\|_{L^2}^2 = \frac{|c|^2}{1 - |p|^2} + |b|^2 \ , \\ M &= (Du|u) = \frac{|c|^2}{(1 - |p|^2)^2} \ , \\ E_\alpha &= \frac{1}{4}\|u\|_{L^4}^4 + \frac{\alpha}{2}|(u|1)|^2 \\ &= \frac{1}{4}\left[|b|^4 + \frac{4|b|^2|c|^2}{1 - |p|^2} + \frac{|c|^4(1 + |p|^2)}{(1 - |p|^2)^3} + \frac{4|c|^2 \mathrm{Re}(bp\bar{c})}{(1 - |p|^2)^2}\right] + \frac{\alpha}{2}|b|^2 \ . \end{aligned}$$

The element $u \in \mathcal{L}(1)$ stays in a compact of $\mathcal{L}(1)$ if and only if $|b| \leq C$, $\frac{1}{C} \leq |c| \leq C$ and $|p| \leq k < 1$ with some constant C and k. Otherwise, due to the formulas of mass Q and momentum M, there exist $t_n \to \infty$ such that $|c(t_n)|$ and $1 - |p(t_n)|^2$ tend to 0 at the same order. Using the formula of Q and E_α, we have

$$|b(t_n)|^2 \to Q, \ \frac{1}{4}|b(t_n)|^4 + \frac{\alpha}{2}|b(t_n)|^2 \to \ E_\alpha \ .$$

Since the limit should be unique,

$$E_\alpha = \frac{1}{4}Q^2 + \frac{\alpha}{2}Q \ .$$

Using the formula of mass and energy, (1.3.2) can be rewritten under coordinates of b, p, c as

$$|b|^2 + \frac{|c|^2|p|^2}{(1 - |p|^2)^2} + 2\mathrm{Re}(\frac{bp\bar{c}}{1 - |p|^2}) = \alpha \tag{1.3.4}$$

simplifying the left hand side, we get

$$\left| b + \frac{\bar{p}c}{1 - |p|^2} \right| = \sqrt{\alpha} \ . \tag{1.3.5}$$

Now, we turn to prove that (1.3.5) is sufficient to cause the exponential growth of Sobolev norms. Writing as before

$$u(t) = b(t) + \frac{c(t)z}{1 - p(t)z} \ ,$$

then the terms $\partial_t u$, $\Pi(|u|^2u)$, $(u|1)$ can be represented as linear combinations of 1, $\frac{z}{1-pz}$ and $\frac{z^2}{(1-pz)^2}$,

$$\begin{cases} \partial_t u = \partial_t b + \partial_t c \dfrac{z}{1-pz} + c\partial_t p \dfrac{z^2}{(1-pz)^2} \ , \\ \Pi(|u|^2u) = |b|^2 b + \dfrac{2b|c|^2}{1-|p|^2} + \dfrac{|c|^2 c\overline{p}}{(1-|p|^2)^2} \\ \qquad + \Big[2|b|^2 c + \dfrac{2b|c|^2 p}{1-|p|^2} + \dfrac{1}{(1-|p|^2)2}|c|^2 c\Big] \dfrac{z}{1-pz} \\ \qquad + \Big[c^2\overline{b} + \dfrac{|c|^2 cp}{1-|p|^2}\Big] \dfrac{z^2}{(1-pz)^2} \ , \\ (u \mid 1) = b \ . \end{cases}$$

then (1.1.6) reads

$$\begin{cases} i\partial_t b = |b|^2 b + \dfrac{2b|c|^2}{1-|p|^2} + \dfrac{|c|^2 c\overline{p}}{(1-|p|^2)^2} + \alpha b \ , \\ i\partial_t c = 2|b|^2 c + \dfrac{2b|c|^2 p}{1-|p|^2} + \dfrac{|c|^2 c}{(1-|p|^2)^2} \ , \\ i\partial_t p = c\overline{b} + \dfrac{|c|^2 p}{1-|p|^2} \ . \end{cases} \tag{1.3.6}$$

Using the second equation of (1.3.6), we gain

$$\frac{d(|c|)^2}{dt} = \frac{4|c|^2}{1-|p|^2}\mathrm{Im}(bp\overline{c}) \ , \tag{1.3.7}$$

by applying (1.3.4) and the formulas of Q , M, we have

$$\begin{aligned} \Big(\frac{\mathrm{Im}(bp\overline{c})}{1-|p|^2}\Big)^2 &= \frac{|bp\overline{c}|^2}{(1-|p|^2)^2} - \Big(\frac{\mathrm{Re}(bp\overline{c})}{1-|p|^2}\Big)^2 \\ &= \frac{|b|^2|c|^2}{(1-|p|^2)^2} - \frac{|b|^2|c|^2}{1-|p|^2} - \frac{1}{4}\Big(\alpha - |b|^2 - \frac{|c|^2}{(1-|p|^2)^2} + \frac{|c|^2}{1-|p|^2}\Big)^2 \\ &= M(Q - \frac{|c|^2}{1-|p|^2}) - \frac{|c|^2}{1-|p|^2}(Q - \frac{|c|^2}{1-|p|^2}) - \frac{1}{4}\Big(\alpha - Q - M + 2\frac{|c|^2}{1-|p|^2}\Big)^2 \\ &= M(Q - |c|\sqrt{M}) - |c|\sqrt{M}(Q - |c|\sqrt{M}) - \frac{1}{4}(\alpha - Q - M + 2|c|\sqrt{M})^2 \\ &= -\alpha\sqrt{M}|c| + QM - \frac{1}{4}(\alpha - Q - M)^2 \ . \end{aligned}$$

Thus

$$\frac{d|c|}{dt} = \pm|c|\sqrt{-4\alpha\sqrt{M}|c| + 4QM - (\alpha - M - Q)^2} \ . \tag{1.3.8}$$

with $0 \le |c| \le \frac{4QM-(\alpha-M-Q)^2}{4\alpha\sqrt{M}}$. We change variables by

$$y := \sqrt{-4\alpha\sqrt{M}|c| + 4QM - (\alpha - M - Q)^2} \ ,$$

then

$$\begin{cases} y' = \pm\frac{1}{2}(\beta - y^2) \ , \beta = 4QM - (\alpha - M - Q)^2 \\ 0 \le y \le \sqrt{\beta} \ . \end{cases} \tag{1.3.9}$$

We first deal with the case that there exists t_0, such that $y'(t_0) > 0$, here we only discuss the case $t > t_0$ for example. Then

$$y' = \frac{1}{2}(\beta - y^2) \ , 0 \le y(t_0) < \sqrt{\beta} \ ,$$

we may solve this equation explicitly,

$$y(t) = \sqrt{\beta}\frac{1 - \mathrm{e}^{\gamma + \sqrt{\beta}(t-t_0)}}{1 + \mathrm{e}^{\gamma + \sqrt{\beta}(t-t_0)}} \ , \gamma = \ln \frac{\sqrt{\beta} + y(t_0)}{\sqrt{\beta} - y(t_0)} \ge 0 \ ,$$

then

$$|c|(t) = \frac{\beta}{4\alpha\sqrt{M}} \frac{4\mathrm{e}^{\sqrt{\beta}(t-t_0)+\gamma}}{(1 + \mathrm{e}^{\sqrt{\beta}(t-t_0)+\gamma})^2} \ . \tag{1.3.10}$$

If $y'(t_0) < 0$, then

$$y' = -\frac{1}{2}(\beta - y^2) \ , 0 < y(t_0) \le \sqrt{\beta} \ ,$$

then we may solve the above equation explicitly,

$$y(t) = \sqrt{\beta}\frac{1 - \mathrm{e}^{\gamma - \sqrt{\beta}(t-t_0)}}{1 + \mathrm{e}^{\gamma - \sqrt{\beta}(t-t_0)}} \ .$$

Thus there exists $t_1 < \infty$ such that $y(t_1) = 0$, and for $t > t_1$, y satisfies

$$y' = \frac{1}{2}(\beta - y^2) \ , y(t_1) = 0 \ .$$

Then we are back to the first case, and will get a similar decay formula as (1.3.10).

Notice that $\widehat{u}(k,t) = cp^{k-1}$ for $k \ge 1$, using Fourier expansion, we obtain, as $|p|$ approaches 1,

$$\|u\|_{H^s}^2 \simeq \frac{|c|^2}{(1 - |p|^2)^{2s+1}} \ .$$

Since $M(u) = \frac{|c|^2}{(1-|p|^2)^2} =$ constant, we get $\|u\|_{H^s}^2 \simeq |c|^{-(2s-1)} \simeq \mathrm{e}^{C(2s-1)|t|}$, which has an exponential growth as $s > \frac{1}{2}$. The proof is complete. □

Corollary 1.3.1. *We do not have the growth of H^s norms for small data in $\mathcal{L}(1)$. In other words, if $\|u(0)\|_{H_+^{\frac{1}{2}}} << \sqrt{\alpha}$, the higher Sobolev norm will never grow to infinity.*

Démonstration. $\|u(0)\|_{H_+^{\frac{1}{2}}} << \sqrt{\alpha}$, then

$$\left| b + \frac{c\overline{p}}{1 - |p|^2} \right| \le \sqrt{Q} + \sqrt{M} \lesssim \|u(0)\|_{H_+^{\frac{1}{2}}} << \sqrt{\alpha} \ .$$

According to the necessary and sufficient condition (1.3.5), there is no norm explosion. □

Remark 1.3.3. *For the case $\alpha = 0$, Gérard and Grellier [11, Corollary 5] got the following instability of H^s norms. Consider a family of Cauchy data given as*

$$u_0^\varepsilon = z + \varepsilon,\ \varepsilon \in \mathbb{C}\ ,$$

then for any ε, there exists t_ε, $t_\varepsilon \to \infty$ as $\varepsilon \to 0$, then the higher Sobolev norm of the corresponding solution tends to infinity as as $\varepsilon \to 0$, more precisely,

$$\|u^\varepsilon(t^\varepsilon)\|_{H^s} \simeq t_\varepsilon^{-(2s-1)},\ s > \frac{1}{2}\ .$$

However, we do not have such an instability result for $\alpha > 0$. In fact, using the theorem 1.3.1, we know there exists a constant $C = C(\alpha, \varepsilon)$ such that for any $\varepsilon \neq \alpha$,

$$\sup_{t\in\mathbb{R}} \|u^\varepsilon(t)\|_{H^s} < C(\alpha, \varepsilon)\ .$$

Now, we give an example to display the energy cascade in Theorem 1.3.1.

Theorem 1.3.2. *Given $\alpha > 0$.*

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1)\ , \\ u|_{t=0} = z + \sqrt{\alpha}, \quad z \in \mathbb{S}^1\ . \end{cases} \tag{1.3.11}$$

For all $s > \frac{1}{2}$, the above equation is globally well-posed in H^s and the solution satisfies

$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{(2s-1)\sqrt{\alpha}t},\ t \to \infty\ .$$

Démonstration. Firstly, since $u_0 = z + \sqrt{\alpha} \in \mathcal{L}(1)$, then the corresponding solution $u(t) \in \mathcal{L}(1)$. As we did before, the solution $u(t)$ can be represented as $u(t) = b(t) + \frac{c(t)}{1-p(t)z}$, and admits the following conserved quantities

$$Q = 1 + \alpha,\ M = 1,\ E_\alpha = \frac{1}{4}(1+\alpha)(1+3\alpha)\ .$$

By the proof of Theorem 1.3.1,

$$(\frac{d}{dt}|c|)^2 = 4\alpha|c|^2(1-|c|)\ .$$

Together with the initial condition $|c|(0) = 1$, we get for $t > 0$ (same strategy for $t < 0$),

$$\frac{d}{dt}|c| = -2\sqrt{\alpha}|c|\sqrt{1-|c|}\ . \tag{1.3.12}$$

$$|c|(t) = \frac{4\mathrm{e}^{2\sqrt{\alpha}t}}{(1+\mathrm{e}^{2\sqrt{\alpha}t})^2}\ .$$

By (1.3.5), we can get

$$\mathrm{Re}(bp\bar{c}) = |c|^2 - |c|\ ,$$

and by (1.3.7) and (1.3.12), we have

$$\mathrm{Im}(bp\bar{c}) = -\sqrt{\alpha}|c|\sqrt{1-|c|}\ ,$$

so

$$bp\bar{c} = \mathrm{Re}(bp\bar{c}) + i\mathrm{Im}(bp\bar{c}) = |c|^2 - |c| - i\sqrt{\alpha}|c|\sqrt{1-|c|}\ .$$

The second equation of (1.3.6) can be simplified as follows,

$$\begin{cases} i\partial_t c = (1+2\alpha - 2i\sqrt{\alpha}\sqrt{1-|c|})c\ , \\ c(0) = 1\ . \end{cases}$$

Then

$$c(t) = \frac{4\mathrm{e}^{2\sqrt{\alpha}t}}{(1+\mathrm{e}^{2\sqrt{\alpha}t})^2}\mathrm{e}^{-i(1+2\alpha)t}\ . \tag{1.3.13}$$

Now, we turn to calculate b and p, in fact, we only need to calculate their angles. Let us denote

$$b = |b|\mathrm{e}^{i\theta(t)} = \sqrt{1+\alpha-|c|}\mathrm{e}^{i\theta(t)}\ ,\ p = |p|\mathrm{e}^{i\sigma(t)} = \sqrt{1-|c|}\mathrm{e}^{i\sigma(t)}\ ,$$

then using the differential equation on p, we get

$$\partial_t\sigma|p| = |c||p| + \mathrm{Re}(c\bar{b}\mathrm{e}^{-i\sigma}) = |c||p| + \mathrm{Re}(\frac{c\bar{b}\bar{p}}{|p|}) = |c||p| + \frac{1}{|p|}(|c|^2-|c|) = 0\ ,$$

which means

$$\sigma(t) = \sigma(0)\ .$$

Since

$$\begin{aligned} bp &= \frac{c(bp\bar{c})}{|c|^2} = (|c|-1-i\sqrt{\alpha}\sqrt{1-|c|})\mathrm{e}^{-i(1+2\alpha)t} \\ &= \sqrt{(1+\alpha-|c|)(1-|c|)}(-\frac{\sqrt{1-|c|}}{\sqrt{1+\alpha-|c|}} - i\frac{\sqrt{\alpha}}{\sqrt{1+\alpha-|c|}})\mathrm{e}^{-i(1+2\alpha)t}\ , \\ \mathrm{e}^{i(\theta+\sigma)} &= (-\frac{\sqrt{1-|c|}}{\sqrt{1+\alpha-|c|}} - i\frac{\sqrt{\alpha}}{\sqrt{1+\alpha-|c|}})\mathrm{e}^{-i(1+2\alpha)t}\ , \end{aligned}$$

and $\mathrm{e}^{i\theta(0)} = 1$, thus we get

$$\mathrm{e}^{i\sigma(t)} = \mathrm{e}^{i\sigma(0)} = \mathrm{e}^{i(\sigma(0)+\theta(0))} = -i\ ,$$

then

$$\mathrm{e}^{i\theta(t)} = (-i\frac{\sqrt{1-|c|}}{\sqrt{1+\alpha-|c|}} + \frac{\sqrt{\alpha}}{\sqrt{1+\alpha-|c|}})\mathrm{e}^{-i(1+2\alpha)t}\ .$$

Finally, we have

$$\begin{aligned} p(t) &= -i\sqrt{1-|c|} = -i\frac{\mathrm{e}^{2\sqrt{\alpha}t}-1}{\mathrm{e}^{2\sqrt{\alpha}t}+1}\ , \\ b(t) &= (\sqrt{\alpha} - i\frac{\mathrm{e}^{2\sqrt{\alpha}t}-1}{\mathrm{e}^{2\sqrt{\alpha}t}+1})\mathrm{e}^{-i(1+2\alpha)t}\ . \end{aligned} \tag{1.3.14}$$

Now, we get the explicit formula for the solution $u(t) = b(t) + \frac{c(t)z}{1-p(t)z}$,

$$\begin{cases} b(t) = (\sqrt{\alpha} - i\dfrac{\mathrm{e}^{2\sqrt{\alpha}t} - 1}{\mathrm{e}^{2\sqrt{\alpha}t} + 1})\mathrm{e}^{-i(1+2\alpha)t} \\ c(t) = \dfrac{4\mathrm{e}^{2\sqrt{\alpha}t}}{(1+\mathrm{e}^{2\sqrt{\alpha}t})^2}\mathrm{e}^{-i(1+2\alpha)t} \ , \\ p(t) = -i\dfrac{\mathrm{e}^{2\sqrt{\alpha}t} - 1}{\mathrm{e}^{2\sqrt{\alpha}t} + 1} \ . \end{cases} \tag{1.3.15}$$

In this case, $M(u) = \frac{|c|^2}{(1-|p|^2)^2} = 1$ and we get for $t \to +\infty$,

$$\|u(t)\|_{H^s}^2 \simeq |c|^{-(2s-1)} \simeq C\mathrm{e}^{2(2s-1)\sqrt{\alpha}t} \ .$$

$\square$

Remark 1.3.4. *One can illustrate this instability of Sobolev norms from the viewpoint of transfer of energy to high frequencies. The Fourier coefficients for* $u = b + \frac{cz}{1-pz}$ *are*

$$\widehat{u}(k) = c(t)p(t)^{k-1}, \ \forall k \geq 1 \ .$$

Then

$$M(u) = 1 = \sum_{k\geq 1} |k||\widehat{u}(k)|^2 = \sum_{k\geq 1} |k||c(t)|^2|p(t)|^{2(k-1)} \ .$$

With (1.3.15), *we have*

$$\sum_{k\geq 1} \Big|\frac{1-\mathrm{e}^{-2\sqrt{\alpha}t}}{1+\mathrm{e}^{-2\sqrt{\alpha}t}}\Big|^{2k} \frac{16|k|}{|(1+\mathrm{e}^{-2\sqrt{\alpha}t})(1-\mathrm{e}^{-2\sqrt{\alpha}t})|^2} = 1 \ .$$

As $t \to \infty$, *we get*

$$\sum_{k\geq 1} 4|k|\mathrm{e}^{-2\sqrt{\alpha}t} \exp(-4|k|\mathrm{e}^{-2\sqrt{\alpha}t}) \sim \frac{1}{4} \ ,$$

so the main part of the summation is on the k*'s satisfying*

$$|k| \sim \mathrm{e}^{2\sqrt{\alpha}t} \ .$$

So as time becomes larger, the main part of the energy concentrates on the Fourier modes as large as $\mathrm{e}^{2\sqrt{\alpha}t}$.

On the other hand, from the viewpoint of the space variable, we find that as time grows to infinity, the energy will concentrate on one point. In fact, rewrite $z = \mathrm{e}^{ix}$, *then*

$$\begin{aligned} \Big|u(t,x) - \sqrt{\alpha} - i\frac{1-\mathrm{e}^{-2\sqrt{\alpha}t}}{1+\mathrm{e}^{-2\sqrt{\alpha}t}}\Big| &= \frac{|c(t)|}{|1-p(t)z|} = \frac{1-|p(t)|^2}{|1-p(t)z|} \sim \frac{1-|p(t)|}{|1-p(t)z|} \\ &\sim \frac{1}{\sqrt{2(\mathrm{e}^{4\sqrt{\alpha}t}-1)(1-\sin x)+4}} \\ &\to 0 \textit{ if and only if } x \neq \frac{\pi}{2}, \ t \to \infty \ . \end{aligned}$$

Therefore, as time tends to infinity, the value of $|u|$ will concentrate on the point $i \in \mathbb{S}^1$.

Moreover, this example shows that the radius of analyticity of the solution of equation (1.1.6) may decay exponentially. This shows the optimality of the result in the recent work [18].

Now, let us turn to the case $\alpha < 0$.

Theorem 1.3.3. *In the case $\alpha < 0$, for any given initial data $u_0 \in \mathcal{L}(1)$, let $u = \frac{az+b}{1-pz}$ be the corresponding solution of (1.1.6). Then there exists a constant $C = C(\alpha)$, such that*

$$\forall t,\ \|u(t)\|_{H^s} < C,\ s \geq \frac{1}{2}\ ,$$

the constant $C > 0$ is uniform for u_0 in a compact subset of $\mathcal{L}(1)$.

Démonstration. We prove this theorem by contradiction. If $u(t_n)$ would leave any compact subset of $\mathcal{L}(1)$, then the Theorem 1.3.1 would lead to (1.3.2), or equivalently to the following equality,

$$\|u_0\|_{L^2}^4 - \|u_0\|_{L^4}^4 = 2\alpha(|(u_0|1)|^2 - \|u_0\|_{L^2}^2)\ .$$

Via the Cauchy-Schwarz inequality and $\alpha < 0$, we get

$$\|u_0\|_{L^2} = \|u_0\|_{L^4} \text{ and } |(u_0|1)| = \|u_0\|_{L^2}\ ,$$

then u_0 should be a constant, which contradicts the fact that $u_0 \in \mathcal{L}(1)$. □

1.4 Further studies and open problems

In this chapter, we just considered the data on the 3-(complex) dimensional manifold

$$\mathcal{L}(1) := \{u : \ \mathrm{rk} K_u = 1\}\ .$$

It is of course natural to consider the higher dimensional case, which will be probably much more complicated. Since we have also got enough conservation laws for the case $\mathrm{rk}K_u = 2$, we have a conjecture that the system stays completely integrable for $\mathrm{rk}K_u \geq 2$. It would be interesting to know how the results of this paper extend to this bigger phase space. In particular, do small data generate large time blow up of high Sobolev norms ?

Chapitre 2

The cubic Szegő equation with a linear perturbation

2.1 Introduction

The purpose of this chapter is to study the following Hamiltonian system,

$$i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1) \ , \ x \in \mathbb{S}^1 \ , \ t \in \mathbb{R} \ , \ \alpha \in \mathbb{R} \ . \tag{2.1.1}$$

where the operator Π is defined as a projector onto the non-negative frequencies, which is called the Szegő projector. When $\alpha = 0$, the equation above turns out to be the cubic Szegő equation,

$$i\partial_t u = \Pi(|u|^2 u) \ , \tag{2.1.2}$$

which was introduced by P. Gérard and S. Grellier as an important mathematical model of the completely integrable systems and non-dispersive dynamics [11, 13]. For $\alpha \neq 0$, by changing variables as $u = \sqrt{|\alpha|}\tilde{u}(|\alpha|t)$, then $\tilde{u}$ satisfies

$$i\partial_t \tilde{u} = \Pi(|\tilde{u}|^2 \tilde{u}) + \operatorname{sgn}(\alpha)(\tilde{u}|1) \ . \tag{2.1.3}$$

Thus our target equation with $\alpha \neq 0$ becomes

$$i\partial_t u = \Pi(|u|^2 u) \pm (u|1) \ . \tag{2.1.4}$$

2.1.1 Lax Pair structure

Thanks to the Lax pairs for the cubic Szegő equation (2.1.2) [13], we are able to find a Lax pair for (2.1.1). To introduce the Lax pair structure, let us first define some useful operators and notation. For $X \subset \mathcal{D}'(\mathbb{S}^1)$, we denote

$$X_+(\mathbb{S}^1) := \left\{ u(\mathrm{e}^{i\theta}) \in X, \ u(\mathrm{e}^{i\theta}) = \sum_{k\geq 0} \hat{u}(k)\mathrm{e}^{ik\theta} \ \right\} \ . \tag{2.1.5}$$

For example, L^2_+ denotes the Hardy space of L^2 functions which extend to the unit disc $\mathbb{D} = \{z \in \mathbb{C}, \ |z| < 1\}$ as holomorphic functions

$$u(z) = \sum_{k\geq 0} \hat{u}(k) z^k, \ \sum_{k\geq 0} |\hat{u}(k)|^2 < \infty \ . \tag{2.1.6}$$

Then the Szegő operator Π is an orthogonal projector $L^2(\mathbb{S}^1) \to L^2_+(\mathbb{S}^1)$.

Now, we are to define a Hankel operator and a Toeplitz operator. By a Hankel operator we mean a bounded operator Γ on the sequence space ℓ^2 which has a Hankel matrix in the standard basis $\{e_j\}_{j\geq 0}$,

$$(\Gamma e_j,\ e_k\ ) = \gamma_{j+k},\ j,\ k \geq 0\ , \tag{2.1.7}$$

where $\{\gamma_j\}_{j\geq 0}$ is a sequence of complex numbers. More backgrounds on the Hankel operators can be found in [41].

Let S be the shift operator on ℓ^2,

$$Se_j = e_{j+1},\ j \geq 0\ .$$

It is easy to show that a bounded operator Γ on ℓ^2 is a Hankel operator if and only if

$$S^*\Gamma = \Gamma S\ . \tag{2.1.8}$$

Definition 2.1.1. *For any given $u \in H^{\frac{1}{2}}_+(\mathbb{S}^1)$, $b \in L^\infty(\mathbb{S}^1)$, we define two operators $H_u,\ T_b:\ L^2_+ \to L^2_+$ as follows. For any $h \in L^2_+$,*

$$H_u(h) = \Pi(u\bar{h})\ , \tag{2.1.9}$$
$$T_b(h) = \Pi(bh)\ . \tag{2.1.10}$$

Notice that H_u is $\mathbb{C}-$antilinear and symmetric with respect to the real scalar product $\mathrm{Re}(u|v)$. In fact, it satisfies

$$(H_u(h_1)|h_2) = (H_u(h_2)|h_1)\ .$$

T_b is $\mathbb{C}-$linear and is self-adjoint if and only if b is real-valued.

Moreover, H_u is a Hankel operator. Indeed, it is given in terms of Fourier coefficients by

$$\widehat{H_u(h)}(k) = \sum_{\ell\geq 0} \hat{u}(k+\ell)\overline{\hat{h}(\ell)}\ ,$$

then

$$\begin{aligned} S^*H_u(h) &= \sum_{k,\ell\geq 0} \hat{u}(k+\ell)\overline{\hat{h}(\ell)}S^*e_k = \sum_{k,\ell\geq 0} \hat{u}(k+\ell+1)\overline{\hat{h}(\ell)}e_k\ , \\ H_uSh &= \sum_{k\geq \ell,\ell\geq 0} \hat{u}(k)e_k\overline{\hat{h}(\ell)e_{\ell+1}} = \sum_{k,\ell\geq 0} \hat{u}(k+\ell+1)\overline{\hat{h}(\ell)}e_k\ , \end{aligned}$$

which means $S^*H_u = H_uS$, thus H_u is a Hankel operator. We may also represent T_b in terms of Fourier coefficients,

$$\widehat{T_b(h)}(k) = \sum_{\ell\geq 0} \hat{b}(k-\ell)\hat{h}(\ell)\ ,$$

then its matrix representation, in the basis $e_k, k \geq 0$, has constant diagonals, T_b is a Toeplitz operator.

We now define another operator $K_u := T_z^* H_u$. In fact T_z is exactly the shift operator S as above, we then call K_u the shifted Hankel operator, which satisfying the following identity

$$K_u^2 = H_u^2 - (\cdot \mid u)u \ . \tag{2.1.11}$$

Using the operators above, Gérard and Grellier found two Lax pairs for the Szegő equation (2.1.2).

Theorem 2.1.1. *[11, Theorem 3.1] Let $u \in C(R, H_+^s(\mathbb{S}^1))$ for some $s > 1/2$. The cubic Szegő equation (2.1.2) has two Lax pairs (H_u, B_u) and (K_u, C_u), namely, if u solves (2.1.2), then*

$$\frac{dH_u}{dt} = [B_u, H_u] \ , \frac{dK_u}{dt} = [C_u, K_u] \ , \tag{2.1.12}$$

where

$$B_u := \frac{i}{2} H_u^2 - iT_{|u|^2} \ , \ C_u = \frac{i}{2} K_u^2 - iT_{|u|^2} \ .$$

For $\alpha \neq 0$, the perturbed Szegő equation (2.1.1) is globally well-posed and by simple calculus, we find that (H_u, B_u) is no longer a Lax pair, in fact,

$$\frac{dH_u}{dt} = [B_u, H_u] - i\alpha(u|1)H_1 \ . \tag{2.1.13}$$

Fortunately, (K_u, C_u) is still a Lax pair.

Theorem 2.1.2. *[50] Given $u_0 \in H_+^{\frac{1}{2}}(\mathbb{S}^1)$, there exists a unique global solution $u \in C(\mathbb{R}; H_+^{\frac{1}{2}})$ of (2.1.1) with u_0 as the initial condition. Moreover, if $u_0 \in H_+^s(\mathbb{S}^1)$ for some $s > \frac{1}{2}$, then $u \in C^\infty(\mathbb{R}; H_+^s)$. Furthermore, the perturbed Szegő equation (2.1.1) has a Lax pair (K_u, C_u), namely, if u solves (2.1.1), then*

$$\frac{dK_u}{dt} = [C_u, K_u] \ . \tag{2.1.14}$$

An important consequence of this structure is that, if u is a solution of (2.1.1), then $K_{u(t)}$ is unitarily equivalent to K_{u_0}. In particular, the spectrum of the $\mathbb{C}$-linear positive self-adjoint trace class operator K_u^2 is conserved by the evolution.

Denote

$$\mathcal{L}(N) := \left\{ u : \mathrm{rk}(K_u) = N, N \in \mathbb{N}^+ \right\} \ . \tag{2.1.15}$$

Thanks to the Lax pair structure, the manifolds $\mathcal{L}(N)$ are invariant under the flow of (2.1.1). Moreover, they turn out to be spaces of rational functions as in the following Kronecker type theorem.

Theorem 2.1.3. *[50] $u \in \mathcal{L}(N)$ if and only if $u(z) = \frac{A(z)}{B(z)}$ is a rational function with*

$$A, B \in \mathbb{C}_N[z], A \wedge B = 1, \deg(A) = N \text{ or } \deg(B) = N, B^{-1}(\{0\}) \cap \overline{\mathbb{D}} = \emptyset \ ,$$

where $A \wedge B = 1$ means A and B have no common factors.

Our main objective of the study on this mathematical model (2.1.1) is on the large time unboundedness of the solution. This general question of existence of unbounded Sobolev trajectories comes back to [4], and was addressed by several authors for various Hamiltonian PDEs, see e.g. [7, 12, 22, 23, 24, 21, 25, 26, 27, 30, 44]. We have already considered the case with initial data $u_0 \in \mathcal{L}(1)$ and found that

Theorem 2.1.4. *[50] Let u be a solution to the α–Szegő equation,*

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1) \ , \ \alpha = \mathbb{R} \ , \\ u(0,x) = u_0(x) \in \mathcal{L}(1) \ . \end{cases} \tag{2.1.16}$$

For $\alpha < 0$, the Sobolev norm of the solution will stay bounded, uniform if u_0 is in some compact subset of $\mathcal{L}(1)$,

$$\|u(t)\|_{H^s} \leq C \ , \ C \text{ does not depend on time } t \ , \ s \geq 0 \ .$$

For $\alpha > 0$, the solution u of the α–Szegő equation has an exponential-on-time Sobolev norm growth,

$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{C_s|t|} \ , \ s > \frac{1}{2} \ , \ C_s > 0 \ , \ |t| \to \infty \ , \tag{2.1.17}$$

if and only if

$$E_\alpha = \frac{1}{4}Q^2 + \frac{1}{2}Q, \tag{2.1.18}$$

with E_α and Q as the two conserved quantities, energy and mass.

2.1.2 Main results

We continue our studies on the cubic Szegő equation with a linear perturbation (2.1.1) on the circle $\mathbb{S}^1$ with more general initial data $u_0 \in \mathcal{L}(N)$ for any $N \in \mathbb{N}^+$.

Firstly, the system is integrable since there are a large amount of conservation laws which comes from the Lax pair structure(2.1.14).

Theorem 2.1.5. *Let $u(t,x)$ be a solution of (2.1.1). For every Borel function f on $\mathbb{R}$, the following quantity*

$$L_f(u) := \Big(f(K_u^2)u|u\Big) - \alpha\Big(f(K_u^2)1|1\Big)$$

is conserved.

Let σ_k^2 be an eigenvalue of K_u^2, and f be the characteristic function of the singleton $\{\sigma_k^2\}$, then

$$\ell_k(u) := \|u_k'\|^2 - \alpha\|v_k'\|^2$$

is conserved, where u_k', v_k' are the projections of u and 1 onto $\ker(K_u^2 - \sigma_k^2)$, and $\|\cdot\|$ denotes the L^2–norm on the circle. Generically, on the $2N+1$–dimensional complex manifold $\mathcal{L}(N)$, we have $2N+1$ linearly independent and in involution conservation laws, which are σ_k , $1 \leq k \leq N$ and ℓ_m , $0 \leq m \leq N$. Thus, the system (2.1.1) can be approximated by a sequence of systems of finite dimension which are completely integrable in the Liouville sense.

Secondly, we prove the existence of unbounded trajectories for data in $\mathcal{L}(N)$ for any arbitrary $N \in \mathbb{N}^+$. One way to capture the unbounded trajectories of solutions is via the motion of singular values of H_u^2 and K_u^2. In the case with $\alpha = 0$, all the eigenvalues of H_u^2 and K_u^2 are constants, but the eigenvalues of H_u^2 are no longer constants for $\alpha \neq 0$, which makes the system more complicated.

By studying the motion of singular values of H_u and K_u, we gain that the necessary condition and existence of crossing which means the two closest eigenvalues of H_u touch some eigenvalue of K_u at some finite time. A remarkable observation is that the Blaschke products of K_u never change their $\mathbb{S}^1$ orbits as time goes.

The main result on the large time behaviour of solutions is as below.

Theorem 2.1.6. *Let $u_0 \in \mathcal{L}(N)$ for any $N \in \mathbb{N}^+$.*

If $\alpha < 0$, the trajectory of the solution $u(t)$ of the α–Szegő (2.1.1) stays in a compact subset of $\mathcal{L}(N)$. In other words, the Sobolev norm of the solution $u(t)$ will stay bounded,

$$\|u(t)\|_{H^s} \leq C \ , \ C \textit{ does not depend on time } t \ , \ s \geq 0 \ .$$

While for $\alpha > 0$, there exists $u_0 \in \mathcal{L}(N)$ which leads to a solution with norm explosion at infinity. More precisely,

$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{C_\alpha (2s-1)|t|} \ , \ t \to \infty \ , \ \forall s \geq \frac{1}{2} \ .$$

Remark 2.1.1.

1. *In the case $\alpha = 0$, there are two Lax pairs, the conserved quantities are much simpler, which are the eigenvalues of H_u^2 and K_u^2. While in the case $\alpha \neq 0$, the eigenvalues of H_u^2 are no longer conserved, which makes our system more complicated.*

2. *For the cubic Szegő equation with $\alpha = 0$, Gérard and Grellier [10] have proved there exists a G_δ dense set $\mathfrak{g}$ of initial data in $C_+^\infty := \cap_s H^s$, such that for any $v_0 \in \mathfrak{g}$, there exist sequences of time $\overline{t_n}$ and $\underline{t_n}$, such that the corresponding solution v of the cubic Szegő equation*

$$i\partial_t v = \Pi_+(|v|^2 v) \ , \ v(0) = v_0 \ , \tag{2.1.19}$$

satisfies

$$\forall r > \frac{1}{2} \ , \ \forall M \geq 1 \ , \frac{\|v(\overline{t_n})\|_{H^r}}{|\overline{t_n}|^M} \to \infty \ , \ n \to \infty \ , \tag{2.1.20}$$

while

$$v(\underline{t_n}) \to v_0 \textit{ in } C_+^\infty \ , \ n \to \infty \ . \tag{2.1.21}$$

Here, by considering the rational data in the case $\alpha \neq 0$, we proved the existence of solutions with exponential growth in time rather than $\limsup$.

There is another non dispersive example with norm growth by Oana Pocovnicu [44], who studied the cubic Szegő equation on the line $\mathbb{R}$, and found there exist solutions with Sobolev norms growing polynomially in time as $|t|^{2s-1}$ with $s \geq 1/2$.

3. *For the case $\alpha > 0$, we now have solutions of (2.1.1) with different growths, uniformly bounded, growing in fluctuations with a* $\limsup$ *super-polynomial in time growth, and* *exponential in time growth. Indeed, it is easy to show that $zu(t,z^2)$ is a solution to the α–Szegő equation if $u(t,z)$ solves the cubic Szegő equation* (2.1.19)*. Thus, for the cubic Szegő equation with a linear perturbation* (2.1.1)*, there also exist solutions with such an energy cascade as in* (2.1.20) *and* (2.1.21)*.*

4. *In this chapter, we consider data in $\mathcal{L}(N)$ for any arbitrary $N \in \mathbb{N}^+$. The data we find which lead to a large time norm explosion are very special. An interesting observation is that the equations on u'_k and v'_k look similar to the original α–Szegő equation,*

$$\frac{\partial}{\partial t}\begin{pmatrix} u'_k \\ v'_k \end{pmatrix} = -i\begin{pmatrix} T_{|u|^2} & \alpha(u|1) \\ -(1|u) & T_{|u|^2} - \sigma_k^2 \end{pmatrix} \begin{pmatrix} u'_k \\ v'_k \end{pmatrix} , \tag{2.1.22}$$

which gives us some hope to extend our results to general rational data.

2.1.3 Organization of this chapter

In section 2, we recall the results about the singular values of H_u and K_u [15]. In section 3, we introduce the conservation laws and prove the integrability. In section 4, we study the motion of the singular values of the Hankel operators H_u and K_u, the eigenvalues of H_u move and may touch some eigenvalue of K_u at finite time while the eigenvalues of K_u stay fixed with the corresponding Blaschke products stay in the same orbits. In section 5, we present a necessary condition of the norm explosion, and as a direct consequence, we know that for $\alpha < 0$, the trajectories of the solutions stay in a compact subset. In section 6, we study the norm explosion with $\alpha > 0$ for data in $\mathcal{L}(N)$ with any $N \in \mathbb{N}^+$. We present some open problems in the last section.

2.2 Spectral analysis of the operators H_u and K_u

In this section, let us introduce some notation which will be used frequently and some useful results by Gérard and Grellier in their recent work [15]. We consider $u \in H^s_+(\mathbb{S}^1)$ with $s > \frac{1}{2}$. The Hankel operator H_u is compact by the theorem due to Hartman [28]. Let us introduce the spectral analysis of operators H_u^2 and K_u^2. For any $\tau \geq 0$, we set

$$E_u(\tau) := \ker(H_u^2 - \tau^2 \mathrm{I}),\ F_u(\tau) := \ker(K_u^2 - \tau^2 \mathrm{I}) . \tag{2.2.1}$$

If $\tau > 0$, the $E_u(\tau)$ and $F_u(\tau)$ are finite dimensional with the following properties.

Proposition 2.2.1. *[15] Let $u \in H^s_+(\mathbb{S}^1) \setminus \{0\}$ with $s > 1/2$, and $\tau > 0$ such that*

$$E_u(\tau) \neq \{0\} \quad \text{or} \quad F_u(\tau) \neq \{0\} .$$

Then one of the following properties holds.

1. $\dim E_u(\tau) = \dim F_u(\tau) + 1$, *$u \not\perp E_u(\tau)$, and $F_u(\tau) = E_u(\tau) \cap u^\perp$.*
2. $\dim F_u(\tau) = \dim E_u(\tau) + 1$, *$u \not\perp F_u(\tau)$, and $E_u(\tau) = F_u(\tau) \cap u^\perp$.*

Moreover, if u_ρ and u'_σ denote respectively the orthogonal projections of u onto $E_u(\rho)$, $\rho \in \Sigma_H(u)$, and onto $F_u(\sigma)$, $\sigma \in \Sigma_K(u)$ with

$$\Sigma_H(u) := \{\tau > 0 :\ u \not\perp E_u(\tau)\}, \quad \Sigma_K(u) := \{\tau \geq 0 :\ u \not\perp F_u(\tau)\} .$$

Then

1. $\Sigma_H(u)$ *and* $\Sigma_K(u)$ *are disjoint, with the same cardinality;*
2. *if* $\rho \in \Sigma_H(u)$,
$$u_\rho = \|u_\rho\|^2 \sum_{\sigma \in \Sigma_K(u)} \frac{u'_\sigma}{\rho^2 - \sigma^2} \ , \tag{2.2.2}$$
3. *if* $\sigma \in \Sigma_K(u)$,
$$u'_\sigma = \|u'_\sigma\|^2 \sum_{\rho \in \Sigma_H(u)} \frac{u_\rho}{\rho^2 - \sigma^2} \ . \tag{2.2.3}$$
4. *A non negative number* σ *belongs to* $\Sigma_K(u)$ *if and only if it does not belong to* $\Sigma_H(u)$ *and*
$$\sum_{\rho \in \Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^2 - \sigma^2} = 1 \ . \tag{2.2.4}$$

By the spectral theorem for H_u^2 and K_u^2, which are self-adjoint and compact, we have the following orthogonal decomposition

$$L^2_+ = \overline{\oplus_{\tau > 0} E_u(\tau)} = \overline{\oplus_{\tau \geq 0} F_u(\tau)} \ . \tag{2.2.5}$$

Then we can write u as
$$u = \sum_{\rho \in \Sigma_H(u)} u_\rho = \sum_{\sigma \in \Sigma_K(u)} u'_\sigma \ . \tag{2.2.6}$$

In fact, we are able to describe these two sets $E_u(\tau)$ and $F_u(\tau)$ more explicitly. Recall that a finite Blaschke product of degree k is a rational function of the form

$$\Psi(z) = \mathrm{e}^{-i\psi} \frac{P(z)}{D(z)} \ ,$$

where $\psi \in \mathbb{S}^1$ is called the angle of Ψ and P is a monic polynomial of degree k with all its roots in $\mathbb{D}$, $D(z) = z^k \overline{P}\left(\frac{1}{z}\right)$ as the normalized denominator of Ψ. Here a monic polynomial is a univariate polynomial in which the leading coefficient (the nonzero coefficient of highest degree) is equal to 1. We denote by $\mathcal{B}_k$ the set of all the Blaschke functions of degree k.

Proposition 2.2.2. *[15] Let* $\tau > 0$ *and* $u \in H^s_+(\mathbb{S}^1)$ *with* $s > \frac{1}{2}$.

1. *Assume* $\tau \in \Sigma_H(u)$ *and* $\ell := \dim E_u(\tau) = \dim F_u(\tau) + 1$. *Denote by* u_τ *the orthogonal projection of* u *onto* $E_u(\tau)$. *There exists a Blaschke function* $\Psi_\tau \in \mathcal{B}_{\ell-1}$ *such that*
$$\tau u_\tau = \Psi_\tau H_u(u_\tau) \ ,$$
and if D *denotes the normalized denominator of* Ψ_τ,
$$E_u(\tau) \ = \ \left\{ \frac{f}{D(z)} H_u(u_\tau) \ , \ f \in \mathbb{C}_{\ell-1}[z] \right\} \ , \tag{2.2.7}$$
$$F_u(\tau) \ = \ \left\{ \frac{g}{D(z)} H_u(u_\tau) \ , \ g \in \mathbb{C}_{\ell-2}[z] \right\} , \tag{2.2.8}$$

and for $a = 0, \dots, \ell - 1$, $b = 0, \dots, \ell - 2$,

$$\begin{aligned} H_u\left(\frac{z^a}{D(z)}H_u(u_\tau)\right) &= \tau \mathrm{e}^{-i\psi_\tau}\frac{z^{\ell-a-1}}{D(z)}H_u(u_\tau) \ , && (2.2.9) \\ K_u\left(\frac{z^b}{D(z)}H_u(u_\tau)\right) &= \tau \mathrm{e}^{-i\psi_\tau}\frac{z^{\ell-b-2}}{D(z)}H_u(u_\tau) \ , && (2.2.10) \end{aligned}$$

where ψ_τ denotes the angle of Ψ_τ.

2. *Assume $\tau \in \Sigma_K(u)$ and $m := \dim F_u(\tau) = \dim E_u(\tau) + 1$. Denote by u'_τ the orthogonal projection of u onto $F_u(\tau)$. There exists an inner function $\Psi_\tau \in \mathcal{B}_{m-1}$ such that*

$$K_u(u'_\tau) = \tau \Psi_\tau u'_\tau \ ,$$

and if D denotes the normalized denominator of Ψ_τ,

$$\begin{aligned} F_u(\tau) &= \left\{\frac{f}{D(z)}u'_\tau \ , \ f \in \mathbb{C}_{m-1}[z]\right\} \ , && (2.2.11) \\ E_u(\tau) &= \left\{\frac{zg}{D(z)}u'_\tau \ , \ g \in \mathbb{C}_{m-2}[z]\right\} \ , && (2.2.12) \end{aligned}$$

and, for $a = 0, \dots, m - 1$, $b = 0, \dots, m - 2$,

$$\begin{aligned} K_u\left(\frac{z^a}{D(z)}u'_\tau\right) &= \tau \mathrm{e}^{-i\psi_\tau}\frac{z^{m-a-1}}{D(z)}u'_\tau \ , && (2.2.13) \\ H_u\left(\frac{z^{b+1}}{D(z)}u'_\tau\right) &= \tau \mathrm{e}^{-i\psi_\tau}\frac{z^{m-b-1}}{D(z)}u'_\tau \ , && (2.2.14) \end{aligned}$$

where ψ_τ denotes the angle of Ψ_τ.

We call the elements $\rho_j \in \Sigma_H(u)$ and $\sigma_k \in \Sigma_K(u)$ as the dominant eigenvalues of H_u and K_u respectively. Due to the above achievements, they are in a finite or infinite sequence

$$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots \to 0 \ ,$$

we denote by ℓ_j and m_k as the multiplicities of ρ_j and σ_k respectively. In other words,

$$\dim E_u(\rho_j) = \ell_j \ ,$$
$$\dim F_u(\sigma_k) = m_k \ .$$

Therefore, we may define the dominant ranks of the operators as

$$\mathrm{rk_d}(H_u) := \sum_j \ell_j \ ,$$
$$\mathrm{rk_d}(K_u) := \sum_k m_k \ ,$$

while the ranks of the operators are

$$\mathrm{rk}(H_u) = \sum_j \ell_j + \sum_k (m_k - 1) \ ,$$
$$\mathrm{rk}(K_u) = \sum_j (\ell_j - 1) + \sum_k m_k \ .$$

In this chapter, u_j and u'_k denote the orthogonal projections of u onto $E_u(\rho_j)$ and $F_u(\sigma_k)$ respectively, while v_j and v'_k denote the orthogonal projections of 1 onto $E_u(\rho_j)$ and $F_u(\sigma_k)$. The L^2–norms of u_j and u'_k can be represented in terms of ρ_ℓ's and σ_ℓ's, which was already observed in [14].

Lemma 2.2.1. *Let $u \in H^{\frac{1}{2}}(\mathbb{S}^1)$, $\Sigma_H(u) = \{\rho_j\}$ and $\Sigma_K(u) = \{\sigma_k\}$ with*

$$\rho_1 > \sigma_1 > \rho_2 > \cdots \geq 0 \ .$$

Then

$$\|u_j\|^2 = \frac{\prod\limits_\ell (\rho_j^2 - \sigma_\ell^2)}{\prod\limits_{\ell \neq j} (\rho_j^2 - \rho_\ell^2)} \ , \ \|u'_k\|^2 = \frac{\prod\limits_\ell (\rho_\ell^2 - \sigma_k^2)}{\prod\limits_{\ell \neq k} (\sigma_\ell^2 - \sigma_k^2)} \ .$$

Démonstration. First, we have

$$\big((I - xH_u^2)^{-1} 1 \mid 1\big) = \prod_\ell \frac{1 - x\sigma_\ell^2}{1 - x\rho_\ell^2} \ .$$

In fact, we can rewrite the left hand side as

$$\big((I - xH_u^2)^{-1} 1 \mid 1\big) = \sum_\ell \frac{\|v_\ell\|^2}{1 - x\rho_\ell^2} + 1 - \sum_\ell \|v_\ell\|^2 \ .$$

From Proposition 2.2.2,

$$v_j = \Big(1, \frac{H_u(u_j)}{\|H_u(u_j)\|}\Big) \frac{H_u(u_j)}{\|H_u(u_j)\|} \ ,$$

combined with $\Psi_j H_u(u_j) = \rho_j u_j$, we get

$$\|v_j\|^2 = \frac{|(1, H_u(u_j))|^2}{\|H_u(u_j)\|^2} = \frac{|(H_u(1), u_j)|^2}{\rho_j^2 \|u_j\|^2} = \frac{\|u_j\|^2}{\rho_j^2} \ .$$

Thus

$$\prod_\ell \frac{1 - x\sigma_\ell^2}{1 - x\rho_\ell^2} = \sum_\ell \frac{\|u_\ell\|^2}{\rho_\ell^2 (1 - x\rho_\ell^2)} + 1 - \sum \frac{\|u_\ell\|^2}{\rho_\ell^2} \ .$$

We get, identifying the residues at $x = 1/\rho_j^2$,

$$\|u_j\|^2 = \frac{\prod\limits_\ell (\rho_j^2 - \sigma_\ell^2)}{\prod\limits_{\ell \neq j} (\rho_j^2 - \rho_\ell^2)} \ . \tag{2.2.15}$$

On the other hand, since

$$1 - x((I - xK_u^2)^{-1}u \mid u) = \frac{1}{((I - xH_u^2)^{-1}1 \mid 1)} \ ,$$

then

$$1 - x\Big(\sum_k \frac{\|u_k'\|^2}{1 - x\sigma_k^2} + \|u\|^2 - \sum_k \|u_k'\|^2\Big) = \prod_\ell \frac{1 - x\rho_\ell^2}{1 - x\sigma_\ell^2} \ ,$$

we get, identifying the residues at $x = 1/\sigma_k^2$,

$$\|u_k'\|^2 = \frac{\prod_\ell (\rho_\ell^2 - \sigma_k^2)}{\prod_{\ell \neq k} (\sigma_\ell^2 - \sigma_k^2)} \ . \tag{2.2.16}$$

□

2.3 Conservation laws and the α–Szegő hierarchy

We endow $L^2_+(\mathbb{S}^1)$ with the symplectic form

$$\omega(u, v) = 4\mathrm{Im}(u \mid v) \ .$$

Then (2.1.1) can be rewritten as

$$\partial_t u = X_{E_\alpha}(u) \ , \tag{2.3.1}$$

with X_{E_α} as the Hamiltonian vector field associated to the Hamiltonian function given by

$$E_\alpha(u) := \frac{1}{4}\int_{\mathbb{S}^1} |u|^4 \frac{d\theta}{2\pi} + \frac{\alpha}{2}|(u|1)|^2 \ .$$

The invariance by translation and by multiplication by complex numbers of modulus 1 gives two other formal conservation laws

$$\begin{aligned} &\text{mass :} && Q(u) := \int_{\mathbb{S}^1} |u|^2 \frac{d\theta}{2\pi} = \|u\|^2_{L^2} \ , \\ &\text{momentum :} && M(u) := (Du|u), \ D := -i\partial_\theta = z\partial_z \ . \end{aligned}$$

Moreover, the Lax pair structure leads to the conservation of the eigenvalues of K_u^2. So it is obvious the system is completely integrable for the data in the 3−dimensional complex manifold $\mathcal{L}(1)$. Then what about the general case, for example in $\mathcal{L}(N)$ with arbitrary $N \in \mathbb{N}^+$? Fortunately, we are able to find many more conservation laws by its Lax pair structure (2.1.14). We will then show our system is still completely integrable with data in $\mathcal{L}(N)$ in the Liouville sense.

2.3.1 Conservation laws

Thanks to the Lax pair structure, we are able to find an infinite sequence of conservation laws.

Theorem 2.3.1. *For every Borel function f on $\mathbb{R}$, the following quantity*

$$L_f(u) := \Big(f(K_u^2)u|u\Big) - \alpha\Big(f(K_u^2)1|1\Big)$$

is a conservation law.

Démonstration. From the Lax pair identity

$$\frac{dK_u}{dt} = [C_u, K_u] \ , \ C_u = -iT_{|u|^2} + \frac{i}{2}K_u^2 \ ,$$

we infer

$$\frac{d}{dt}K_u^2 = [-iT_{|u|^2}, K_u^2] \ ,$$

and consequently, for every Borel function f on $\mathbb{R}$,

$$\frac{d}{dt}f(K_u^2) = [-iT_{|u|^2}, f(K_u^2)] \ .$$

On the other hand, the equation reads

$$\frac{d}{dt}u = -iT_{|u|^2}u - i\alpha(u|1) \ .$$

Therefore we obtain

$$\begin{aligned}\frac{d}{dt}\Big(f(K_u^2)u|u\Big) &= \Big([-iT_{|u|^2}, f(K_u^2)]u|u\Big) - i\Big(f(K_u^2)T_{|u|^2}u|u\Big) + i\Big(u|f(K_u^2)T_{|u|^2}u\Big) \\ &\quad -i\alpha(u|1)\Big(f(K_u^2)(1)|u\Big) + i\alpha(1|u)\Big(f(K_u^2)(u)|1\Big) \\ &= -i\alpha\Big[\big(f(K_u^2)(1)|(1|u)u\big) - \big((1|u)u|f(K_u^2)(1)\big)\Big] \ .\end{aligned}$$

Now observe that

$$(1|u)u = H_u^2(1) - K_u^2(1) = T_{|u|^2}(1) - K_u^2(1) \ .$$

We obtain

$$\begin{aligned}\frac{d}{dt}\Big(f(K_u^2)u|u\Big) &= -i\alpha\Big[\big(f(K_u^2)(1)|T_{|u|^2}(1)\big) - \big(T_{|u|^2}(1)|f(K_u^2)(1)\big)\Big] \\ &= \alpha\Big([-iT_{|u|^2}, f(K_u^2)](1)|1\Big) \\ &= \alpha\frac{d}{dt}\Big(f(K_u^2)(1)|1\Big) \ .\end{aligned}$$

□

2.3.2 The α–Szegő hierarchy

By the theorem above, for any $n \in \mathbb{N}$,

$$L_n(u) := \big(K_u^{2n}(u) \mid u\big) - \alpha\big(K_u^{2n}(1) \mid 1\big)$$

is conserved. Then the manifold $\mathcal{L}(N)$ is of $2N+1-$ complex dimension and admits $2N+1$ conservation laws, which are

$$\sigma_k,\ k = 1, \cdots, N \text{ and } L_n(u),\ n = 0, 1, \cdots, N\ .$$

We are to show that all these conservation laws are in involve. Since the σ_k's are constants, it is sufficient to show that all these L_n satisfy the Poisson commutation relations

$$\{L_n, L_m\} = 0\ . \tag{2.3.2}$$

Let us begin with the following lemma which helps us better understand the conserved quantities.

Lemma 2.3.1. *Let $u \in H^{\frac{1}{2}}(\mathbb{S}^1)$, $\Sigma_H(u) = \{\rho_j\}$ and $\Sigma_K(u) = \{\sigma_k\}$ with*

$$\rho_1 > \sigma_1 > \rho_2 > \cdots \geq 0\ .$$

Denote

$$\begin{aligned} J_x(u) &:= \big((1 - xH_u^2)^{-1}(1) \mid 1\big)\ , \\ Z_x(u) &:= \big(1 \mid (1 - xH_u^2)^{-1}(u)\big)\ , \\ F_x(u) &:= \big((1 - xK_u^2)^{-1}(u) \mid u\big)\ , \\ E_x(u) &:= \big((1 - xK_u^2)^{-1}(1) \mid 1\big)\ . \end{aligned}$$

Then

$$F_x(u) = \frac{J_x(u) - 1}{xJ_x(u)}\ , \tag{2.3.3}$$

$$E_x(u) = J_x(u) - x\frac{|Z_x(u)|^2}{J_x(u)}\ . \tag{2.3.4}$$

Démonstration. Recall (2.1.11), for any $f \in H^{\frac{1}{2}}$, we have

$$K_u^2 f = H_u^2 f - (f \mid u)u\ .$$

Denote

$$w(f) = (1 - xH_u^2)^{-1}(f) - (1 - xK_u^2)^{-1}(f)\ , \tag{2.3.5}$$

then

$$\begin{aligned} w(f) &= x\Big(f \mid (1 - xK_u^2)^{-1}(u)\Big)(1 - xH_u^2)^{-1}(u) \\ &= x\Big(f \mid (1 - xH_u^2)^{-1}(u)\Big)(1 - xK_u^2)^{-1}(u)\ . \end{aligned}$$

We may observe the two vectors $(1-xH_u^2)^{-1}(u)$ and $(1-xK_u^2)^{-1}(u)$ are co-linear,

$$(1-xK_u^2)^{-1}(u) = A(1-xH_u^2)^{-1}(u),\ A \in \mathbb{R}\ . \tag{2.3.6}$$

Let us choose $f = u$, then

$$\Big(w(u) \mid u\Big) = (1-A)\Big((1-xH_u^2)^{-1}(u) \mid u\Big) = Ax\Big(u \mid (1-xH_u^2)^{-1}(u)\Big)^2\ . \tag{2.3.7}$$

We are to calculate the factor A. Since

$$\begin{aligned} x\Big(u \mid (1-xH_u^2)^{-1}(u)\Big) &= x\Big(1 \mid (1-xH_u^2)^{-1}H_u^2(1)\Big) \\ &= \sum_{n\geq 0} x^{n+1}\Big(H_u^{2(n+1)}(1) \mid 1\Big) = \sum_{n\geq 0} x^n\Big(H_u^{2n}(1) \mid 1\Big) - 1 = J_x - 1\ , \end{aligned}$$

thus (2.3.7) yields

$$1 - A = (J_x - 1)A\ ,$$

which means

$$A = \frac{1}{J_x}\ .$$

So (2.3.6) turns out to be

$$(1-xK_u^2)^{-1}(u) = \frac{1}{J_x}(1-xH_u^2)^{-1}(u)\ , \tag{2.3.8}$$

then combined with the definition of $w(f)$, we have

$$(1-xH_u^2)^{-1}(f) - (1-xK_u^2)^{-1}(f) = \frac{x}{J_x}\Big(f \mid (1-xH_u^2)^{-1}(u)\Big)(1-xH_u^2)^{-1}(u)\ . \tag{2.3.9}$$

Using the equality (2.3.8),

$$\begin{aligned} F_x = \Big((1-xK_u^2)^{-1}(u) \mid u\Big) &= \frac{1}{J(x)}\Big((1-xH_u^2)^{-1}(u) \mid u\Big) \\ &= \frac{1}{J(x)}\Big((1-xH_u^2)^{-1}H_u^2(1) \mid 1\Big) = \frac{J_x - 1}{xJ_x}\ . \end{aligned}$$

Now, we turn to prove (2.3.4). Use again (2.3.5) with $f = 1$,

$$\begin{aligned} \Big(w(1)|1\Big) &= \Big((1-xH_u^2)^{-1}(1) - (1-xK_u^2)^{-1}(1)|1\Big) = J_x - E_x \\ &= x\Big(1|1-xH_u^2)^{-1}(1)\Big)\Big((1-xK_u^2)^{-1}(1)|1\Big) = x\overline{Z_x}\Big((1-xK_u^2)^{-1}(u)|1\Big)\ , \end{aligned}$$

plugging (2.3.6),

$$\Big((1-xK_u^2)^{-1}(u)|1\Big) = \frac{1}{J_x}\Big((1-xH_u^2)^{-1}(u)|1\Big) = \frac{Z_x}{J_x}\ ,$$

then

$$J_x - E_x = x\overline{Z_x}\frac{Z_x}{J_x} = x\frac{|Z_x|^2}{J_x}\ , \tag{2.3.10}$$

which leads to (2.3.4). $\square$

Now, we are ready to show the following cancellation for the Poisson brackets of the conservation laws.

Theorem 2.3.2. *For any $x \in \mathbb{R}$, we set*

$$L_x(u) = \big((1 - xK_u^2)^{-1}(u) \mid u\big) - \alpha\big((1 - xK_u^2)^{-1}(1) \mid 1\big) \ ,$$

Then $L_x(u(t))$ is conserved, and for every x, y,

$$\{L_x,\ L_y\} = 0 \ . \tag{2.3.11}$$

Démonstration. Using the previous Lemma, we may rewrite

$$L_x = \frac{1}{x}\big(1 - \frac{1}{J_x}\big) - \alpha E_x \ , \tag{2.3.12}$$

with

$$\begin{aligned} J_x(u) &:= \big((1 - xH_u^2)^{-1}(1) \mid 1\big) = 1 + x\big((1 - xH_u^2)^{-1}(u) \mid u\big) \ , \\ E_x(u) &:= \big((1 - xK_u^2)^{-1}(1) \mid 1\big) = J_x(u) - x\frac{|Z_x(u)|^2}{J_x(u)} \ , \\ Z_x(u) &:= \big(1 \mid (1 - xH_u^2)^{-1}(u)\big) \ . \end{aligned}$$

Recall that the identity

$$\{J_x,\ J_y\} = 0 \tag{2.3.13}$$

which was obtained in [11, section 8]. We then have

$$\{L_x,\ L_y\} = \alpha\Big(\frac{y}{xJ_x^2 J_y}\{J_x,\ |Z_y|^2\} - \frac{x}{yJ_y^2 J_x}\{J_y,\ |Z_x|^2\}\Big) + \alpha^2\{E_x,\ E_y\} \ . \tag{2.3.14}$$

Let us first prove that $\{E_x,\ E_y\} = 0$. Notice that

$$E_x(u) = J_x(S^*u) \ , \tag{2.3.15}$$

therefore

$$dE_x(u) \cdot h = dJ_x(S^*u) \cdot (S^*h) = \omega(S^*h, X_{J_x}(S^*u)) = \omega(h, SX_{J_x}(S^*u)) \ .$$

We conclude

$$X_{E_x}(u) = SX_{J_x}(s^*U) \ ,$$

thus

$$\begin{aligned} \{E_x,\ E_y\}(u) &= dE_y(u) \cdot X_{E_x}(u) = dJ_y(S^*u) \cdot S^*SX_{J_x}(S^*u) \\ &= dJ_y(S^*u) \cdot X_{J_x}(S^*u) = \{J_x, J_y\}(S^*u) = 0 \ . \end{aligned}$$

We now show that the coefficient of α in (2.3.14) vanishes identically. It is enough to work on the generic states of $\mathcal{L}(N)$, so we can use the coordinates

$$(\rho_1, \cdots, \rho_{N+1}, \sigma_1, \cdots, \sigma_N, \varphi_1, \cdots, \varphi_{N+1}, \theta_1, \cdots, \theta_N)$$

for which we recall that

$$\omega = \sum_{j=1}^{N+1} d(\frac{\rho_j^2}{2}) \wedge d\varphi_j + \sum_{k=1}^{N} d(\frac{\sigma_k^2}{2}) \wedge d\theta_k \ .$$

Moreover, we have

$$\rho_j u_j = \mathrm{e}^{-i\varphi_j} H_u(u_j) \ ,$$

therefore,

$$Z_x(u) = \sum_{j=1}^{N+1} \frac{\|u_j\|^2}{\rho_j(1 - x\rho_j^2)} \mathrm{e}^{i\varphi_j} \ .$$

Since

$$J_x(u) = \frac{\prod_{k=1}^{N}(1 - x\sigma_k^2)}{\prod_{j=1}^{N+1}(1 - x\rho_j^2)} \ ,$$

we know that

$$\{J_x,\ \varphi_j\} = \frac{2xJ_x}{1 - x\rho_j^2} \ ,$$

and we infer

$$\{J_x,\ Z_y\} = 2ixJ_x \sum_{j=1}^{N+1} \frac{\|u_j\|^2}{\rho_j(1 - x\rho_j^2)(1 - y\rho_j^2)} \mathrm{e}^{i\varphi_j} = \frac{2ixJ_x}{x - y}(xZ_x - yZ_y) \ . \tag{2.3.16}$$

Consequently,

$$\{J_x,\ |Z_y|^2\} = 2\mathrm{Re}(\overline{Z}_y\{J_x,\ Z_y\}) = -\frac{4x^2 J_x}{x - y}\mathrm{Im}(\overline{Z}_y Z_x) \ . \tag{2.3.17}$$

We conclude that

$$\frac{y}{xJ_x^2 J_y}\{J_x,\ |Z_y|^2\} - \frac{x}{yJ_y^2 J_x}\{J_y,\ |Z_x|^2\} = -\frac{4xy}{(x - y)J_x J_y}\big(\mathrm{Im}(\overline{Z}_y Z_x) + \mathrm{Im}(\overline{Z}_x Z_y)\big) = 0 \ . \tag{2.3.18}$$

This completes the proof. □

The last part of this section is devoted to proving that functions $(L_n(u))_{0\le n\le N}$ are generically independent on $\mathcal{L}(N)$. Actually, it is sufficient to discuss the case $|\alpha| << 1$. For α small enough, we may consider the term $\alpha(K_u^{2n}(1)|1)$ as a perturbation, then we only need to study the independence of $F_n := (K_u^{2n}(u)|u)$. Using the formula (2.3.12), for any $0 \le n \le N$,

$$F_n = J_{n+1} - \sum_{\substack{k+j=n \\ j\ge 1, k\ge 0}} F_k J_j \ ,$$

with $J_n = (H_u^{2n}1|1)$. Assume there exists a sequence c_n such that

$$\sum_{n\ge 0} c_n F_n = 0 \ ,$$

we are to prove that $c_n \equiv 0$. Indeed,

$$\sum_{n\geq 0} c_n J_{n+1} - \sum_{n\geq 0} \sum_{\substack{k+j=n \\ j\geq 1, k\geq 0}} c_n F_k J_j = \sum_n (c_n - \sum_{0\leq k\leq N-(n+1)} c_{n+k+1} F_k) J_{n+1} = 0 \ ,$$

since all the J_{n+1} are independent in the complement of a closed subset of measure 0 of $\mathcal{L}(N)$ [11], then for every n,

$$c_n - \sum_{0\leq k\leq N-(n+1)} c_{n+k+1} F_k = 0 \ .$$

Thus $c_N = c_{N-1} = \cdots = c_0 = 0$.

Finally, we now have $2N+1$ linearly independent and in involution conservation laws on a dense open subset of $2N+1$ dimensional complex manifold $\mathcal{L}(N)$, thus our system is completely integrable in the Liouville sense.

2.4 Multiplicity and Blaschke product

Recall the notation in section 2, there are two kinds of eigenvalues of K_u, some are the dominant eigenvalues of K_u, which are denoted as $\sigma_k \in \Sigma_K(u)$, while the others are the dominant eigenvalues of H_u with multiplicities larger than 1. Let us denote $u(t)$ as the solution of the α–Szegő equation with $\alpha \neq 0$. Fortunately, we are able to show that for almost all $t \in \mathbb{R}$, the Hankel operator $H_{u(t)}$ has single dominant eigenvalues with multiplicities equal to 1. In other words, for almost every time $t \in \mathbb{R}$,

$$\mathrm{rk}_{\mathrm{d}} K_{u(t)} = \mathrm{rk} K_{u(t)} = \mathrm{rk} K_{u_0} \ .$$

We call the phenomenon that $H_{u(t_0)}$ has some eigenvalue σ with multiplicity $m \geq 2$ as *crossing at σ at t_0*.

2.4.1 The motion of singular values

Let us first introduce the following Kato-type lemma.

Lemma 2.4.1 (Kato). *Let $P(t)$ be a projector on a Hilbert space $\mathcal{H}$ which is smooth in $t \in I$, then there exists a smooth unitary operator $U(t)$, such that*

$$P(t) = U(t)P(0)U^*(t) \ ,$$

and

$$\frac{d}{dt}U(t) = Q(t)U(t) \ , \ U(0) = \mathrm{Id} \ , \tag{2.4.1}$$

with $Q(t) = [P'(t), P(t)]$.

Démonstration. By simple calculus, we can prove $Q^* = -Q$. Since $P(t)$ is smooth in time, then by the Cauchy theorem for linear ordinary equations, $U(t)$ is well defined. The unitary property of $U(t)$ for every t is a consequence of the anti self-adjointness of Q.

$$\frac{d}{dt}(U(t)^*U(t)) = \frac{d}{dt}U^*U + U^*\frac{d}{dt}U = U^*Q^*U + U^*QU = 0 \ ,$$

thus $U(t)^*U(t) = \mathrm{Id}$. On the other hand,

$$\frac{d}{dt}(U(t)U(t)^*) = \frac{d}{dt}UU^* + U\frac{d}{dt}U^* = QUU^* - UU^*Q \ .$$

It is obvious that Id is a solution to the linear equation $\frac{d}{dt}A = QA - AQ$ with $A(0) = \mathrm{Id}$, using the uniqueness of solutions, we have $U(t)U^*(t) = \mathrm{Id}$. We now prove that $U^*(t)P(t)U(t)$ does not depend on t.

$$\begin{aligned}\frac{d}{dt}(U^*(t)P(t)U(t)) &= \frac{d}{dt}U^*(t)P(t)U(t) + U^*(t)\frac{d}{dt}P(t)U(t) + U^*(t)P(t)\frac{d}{dt}U(t)\\ &= U^*Q^*PU + U^*P'U + U^*PQU\\ &= U^*(P' + [P,Q])U\\ &= U^*(P' - PP' - P'P)U = 0\end{aligned}$$

where we have used $P^2 = P$. This completes the proof. □

If $u_0 \in H^s_+$ with $s > 1$, then the solution $u(t)$ of the α–Szegő equation (2.1.1) is real analytic in t valued in H^s_+. By the Lax pair for K_u, we know that the singular values of K_u are fixed, with constant multiplicities.

Proposition 2.4.1. *Given any initial data $u_0 \in H^s_+$ with $s > 1$, let u be the corresponding solution to the α–Szegő equation. Let $\sigma > 0$ be a singular eigenvalue of K_u with multiplicity m, and write*

$$\sigma_+ > \sigma > \sigma_-$$

where σ_+, σ_- are the closest singular values of K_u, possibly, $\sigma_+ = +\infty$ or $\sigma_- = 0$. Then one of the following two possibilities occurs.

1. *σ is a singular value of $H_{u(t)}$ with multiplicity $m+1$ for every time t, and u is a solution of the cubic Szegő equation* (2.1.2).
2. *There exists a discrete subset T_c of times outside of which the singular values of $H_{u(t)}$ in the interval (σ_-, σ_+) are ρ_1, ρ_2 of multiplicity 1, and σ of multiplicity $m-1$ if $m \geq 2$, with*

   $$\rho_1 > \sigma > \rho_2 \ ,$$

   and ρ_1, ρ_2 are analytic on every interval contained into the complement of T_c.

Démonstration. Let us assume that σ is a singular value of multiplicity $m+1$ of $H_{u(t_0)}$ for some time t_0. Then we may select $\delta > 0$ and $\epsilon > 0$ such that

$$\sigma_+ > \sigma + \epsilon > \sigma > \sigma - \epsilon > \sigma_-$$

such that, for every $t \in [t_0 - \delta, t_0 + \delta]$, $\sigma^2 - \epsilon$ and $\sigma^2 + \epsilon$ are not eigenvalues of $H^2_{u(t)}$. Then we know that $H^2_{u(t)}$ has either σ^2 as an eigenvalue of multiplicity $m+1$, or admits in $(\sigma^2 - \epsilon, \sigma^2 + \epsilon)$ two eigenvalues of multiplicity 1, ρ_1, ρ_2 on both sides of σ. Set

$$P(t) := (2i\pi)^{-1} \int_{C(\sigma^2,\epsilon)} (z\mathrm{Id} - H^2_{u(t)})^{-1} dz \ . \tag{2.4.2}$$

We know that $P(t)$ is an orthogonal projector, depending analytically of $t \in (t_0 - \delta, t_0 + \delta)$, and that $P(t_0)$ is just the projector onto

$$E(t_0) := \ker(H^2_{u(t_0)} - \sigma^2 \mathrm{Id}) \ .$$

Consider the selfadjoint operator

$$A(t) := H^2_{u(t)} P(t)$$

acting on the $(m+1)$-dimensional space $E(t) = \mathrm{Ran} P(t)$. Then its characteristic polynomial is

$$\mathcal{P}(\lambda, t) = (\sigma^2 - \lambda)^{m-1} (\lambda^2 + a(t)\lambda + b(t)) \ ,$$

where a, b are real analytic, real valued functions, such that

$$a^2 - 4b \geq 0 \ .$$

Notice that the condition $a(t)^2 - 4b(t) = 0$ is precisely equivalent to the fact that $H^2_{u(t)}$ has σ^2 as an eigenvalue of multiplicity $m+1$. Since this function is analytic, it is either identically 0, or different from 0 for $0 < |t - t_0| < \delta$ and $\delta > 0$ small enough. Moreover, by the following perturbation analysis, the first condition only occurs if

$$(1|u(t)) = 0$$

for every $t \in (t_0 - \delta, t_0 + \delta)$. Since $(1|u)$ is a real analytic function of t, this would imply that it is identically 0, whence u is a solution of the cubic Szegő equation. We now come back to the perturbation analysis, let $U(t)$ be a unitary operator given as in the Kato-type lemma above, denote

$$B(t) = U^*(t) A(t) U(t) \ ,$$

then

$$B(t_0) = \sigma^2 \mathrm{Id} P(t_0) \ .$$

Let us calculate the derivative of B, we find

$$\frac{d}{dt} B(t) = \frac{d}{dt} \left(U^*(t) H^2_{u(t)} U(t) U^*(t) P(t) U(t) \right) = \frac{d}{dt} \left(U^*(t) H^2_{u(t)} U(t) P(t_0) \right) \ .$$

Since $\frac{d}{dt} U(t) = Q(t) U(t)$ with $Q(t) = [P'(t), P(t)]$, then

$$\frac{d}{dt} B(t) = U^* \Big(\frac{d}{dt} H^2_{u(t)} + [H^2_{u(t)}, Q(t)] \Big) U P(t_0) \ ,$$

using (2.1.13),

$$\frac{d}{dt}H^2_{u(t)} = [B_u, H_u^2] - i\alpha(u|1)H_1H_u + i\alpha(1|u)H_uH_1 \ .$$

For any $h_1, h_2 \in E(t_0)$,

$$([B_u, H_u^2]h_1, h_2) + ([H_u^2, Q]h_1, h_2) = 0 \ ,$$

then

$$(\frac{d}{dt}B(t_0)h_1, h_2) = -i\alpha[(u(t_0)|1)(h_1|u(t_0))(1|h_2) - (1|u(t_0))(u(t_0)|h_2)(h_1|1)] \ .$$

Denote by v, w as the projections onto $E(t_0)$ of 1 and u respectively. If $(u(t_0)|1) \neq 0$, then the corresponding matrix under the base (v, w) turns out to be

$$\begin{pmatrix} -i\alpha(u|1)(v|w) & i\alpha(1|u)\|w\|^2 \\ -i\alpha(u|1)\|v\|^2 & i\alpha(1|u)(w|v) \end{pmatrix}$$

which has a negative determinant if $(u(t_0)|1) \neq 0$. For the case $(u(t_0)|1) = 0$ with $\frac{d^n}{dt^n}(u|1)(t_0) \neq 0$ for some $n \in \mathbb{N}$, we only need to consider $\frac{d^{n+1}}{dt^{n+1}}(B(t))(t_0)$,

$$\left(\frac{d^{n+1}}{dt^{n+1}}B(t_0)h_1, h_2\right) = -i\alpha\left[\left(\frac{d^n}{dt^n}(u|1)\right)(t_0)(h_1|u(t_0))(1|h_2) - \left(\frac{d^n}{dt^n}(1|u)\right)(t_0)(u(t_0)|h_2)(h_1|1)\right] \ ,$$

with any $h_1, h_2 \in E(t_0)$. It is similar as the case $n = 0$. This completes the proof. □

Since $u(t)$ satisfying $(1|u(t)) \equiv 0$ would be a solution of the cubic Szegő equation, which is well studied by Gérard and Grellier [11, 13, 12, 16]. We assume $(1|u)$ is not identically zero in the rest of this article. From the discussion above, we have

Corollary 2.4.1. *The dominant eigenvalues of $H_{u(t)}$ are of multiplicity 1 for almost all $t \in \mathbb{R}$.*

Recall the notation in section 2, by rewriting the conservation laws in Theorem 2.3.1 as

$$L_n := \Big(K_u^{2n}(u) \mid u\Big) - \alpha\Big(K_u^{2n}(1) \mid 1\Big) = \sum_k \sigma_k^{2n}\left(\|u_k'\|^2 - \alpha\|v_k'\|^2\right) \ , \tag{2.4.3}$$

we get the following conserved quantities

$$\ell_k := \|u_k'\|^2 - \alpha\|v_k'\|^2 \ . \tag{2.4.4}$$

Lemma 2.4.2. *Let $\alpha > 0$. If there exists a crossing at σ_k at time $t = t_0$, then $\ell_k < 0$.*

Démonstration. Since there is a crossing at σ_k, then $\sigma_k \in \Sigma_H(u(t_0))$ with multiplicity $m \geq 2$. Then

$$F_u(\sigma_k) = E_u(\sigma_k) \cap u^\perp = \left\{\frac{g}{D}H_u(u_k) : \ g \in \mathbb{C}_{m-2}[z]\right\} \ .$$

Hence, $u_k' = 0$ while $v_k' \neq 0$, since

$$\|v_k'\| = \frac{(1, H_u(u_k))}{\|H_u(u_k)\|} = \frac{\|u_k\|}{\sigma_k} \neq 0 \ . \tag{2.4.5}$$

Thus $\ell_k = \|u_k'\|^2 - \alpha\|v_k'\|^2 < 0$ for $\alpha > 0$. □

Here, we present an example to show the existence of crossing.

Example 2.4.1 (Existence of crossing)**.** *Let $u_0(z) = \frac{z-p}{1-pz}$ with $p \neq 0$ and $|p| < 1$, and u be the corresponding solution to the equation*

$$i\partial_t u = \Pi(|u|^2 u) + (u|1) \ . \tag{2.4.6}$$

It is obvious that $u_0 \in \mathcal{L}(1)$ and $1 \in \Sigma_H(u_0)$ with multiplicity 2, and

$$L_1(u) = \Big(K_u^2(u) \mid u\Big) - \Big(K_u^2(1) \mid 1\Big) = -(1-|p|^2) < 0 \ .$$

Let us represent the Hamiltonian function $E = \frac{1}{4}\|u\|_{L^4}^4 + \frac{1}{2}|(u|1)|^2$ under the coordinates

$$\rho_1, \rho_2, \sigma, \varphi_1, \varphi_2, \theta \ ,$$

$$\begin{aligned} E &= \frac{1}{4}(\rho_1^4 + \rho_2^4 - \sigma^4) \\ &\quad + \frac{1}{2}\frac{\rho_1^2(\rho_1^2-\sigma^2)^2 + \rho_2^2(\sigma^2-\rho_2^2)^2 + 2\rho_1\rho_2(\rho_1^2-\sigma^2)(\sigma^2-\rho_2^2)\cos(\varphi_1-\varphi_2)}{(\rho_1^2-\rho_2^2)^2} \\ &= \frac{1}{4} + \frac{1}{2}|p|^2 \ . \end{aligned}$$

Notice that $\sigma = 1$ and $\rho_1^2 + \rho_2^2 - \sigma^2 = \|u\|_{L^2}^2 = 1$, then $\rho_1^2 + \rho_2^2 = 2$. Set $I = \frac{\rho_1^2 - \rho_2^2}{2}$, $\varphi = \varphi_1 - \varphi_2$, then $\rho_1^2 = 1 + I$ and $\rho_2^2 = 1 - I$, thus we can rewrite E as

$$E = \frac{1}{4}(1 + 2I^2) + \frac{1}{4}(1 + \sqrt{1-I^2}\cos(\varphi)) \ .$$

Thus

$$\begin{aligned} \frac{dI}{dt} &= -2\frac{\partial E}{\partial \varphi} = \frac{1}{2}\sqrt{1-I^2}\sin(\varphi) \\ &= \pm\frac{1}{2}\sqrt{-4I^2 + (8|p|^2-5)I^2 + 4|p|^2(1-|p|^2)} \\ &= \pm\sqrt{(a-I^2)(b+I^2)} \ , \end{aligned}$$

with a, b satisfy

$$\begin{cases} a > 0, b > 0 \ , \\ ab = |p|^2(1-|p|^2) \ , \\ a - b = 2|p|^2 - 5/4 \ . \end{cases}$$

Recall the definition of Jacobi elliptic functions. The incomplete elliptic integral of the first kind F is defined as

$$F(\varphi, k) \equiv \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \ ,$$

then the Jacobi elliptic function sn and cn are defined as follows,

$$sn(F(\varphi,k),k) = \sin\varphi \ ,$$

$$cn(F(\varphi,k),k) = \cos\varphi \ .$$

Then we may solve the above equation,

$$I(t) = \sqrt{a}\mathrm{cn}\left(\sqrt{a+b}(t-t_0) + F\Big(\frac{\pi}{2}, \sqrt{\frac{a}{a+b}}\Big), \sqrt{\frac{a}{a+b}}\right) \ .$$

Therefore, there exists a discrete set of time $0 \in T_c$, such that $I(t) = 0$ for every $t \in T_c$. In other words, crossing happens at $t \in T_c$.

2.4.2 Blaschke product

We aim to show that the Blaschke products $\Psi(t)$ of $K_{u(t)}$ do not change their $\mathbb{S}^1$–orbits as times grows even before or after crossings.

Proposition 2.4.2. *For any open interval Ω contained into the complement of T_c, for any $\sigma_k \in \Sigma_K(u(t))$ with $t \in \Omega$,*

$$K_{u(t)}u_k'(t) = \sigma_k \Psi_k(t) u_k'(t) \ . \tag{2.4.7}$$

Then there exists a function $\psi_k(t) : \Omega \to \mathbb{S}^1$, such that

$$\Psi_k(t) = \mathrm{e}^{i\psi_k(t)}\Psi_k(0) \ , \ t \in \Omega \ . \tag{2.4.8}$$

Démonstration. Differentiating the above equation (2.4.7) and using the Lax pair structure (2.1.14), one obtains

$$[C_u, K_u](u_k') + K_u\left(\frac{du_k'}{dt}\right) = \sigma_k \dot{\Psi}_k u_k' + \sigma_k \Psi_k \frac{du_k'}{dt} \ . \tag{2.4.9}$$

Recall $u_k' = P_k(u)$, where P_k as (2.4.2) by replacing H_u with K_u, then

$$\frac{d}{dt}P_k(t) = [C_u, P_k] \ .$$

Rewriting $\Pi(|u|^2u) = T_{|u|^2}(u) = (iC_u + \frac{1}{2}K_u^2)u$, then the α–Szegő equation (2.1.1) turns out to be

$$\frac{du}{dt} = (C_u - \frac{i}{2}K_u^2)u - i\alpha(u|1) \ ,$$

then

$$\begin{aligned}\frac{du_k'}{dt} &= (\frac{d}{dt}P_k)(u) + P_k(\frac{du}{dt}) \\ &= [C_u, P_k]u + P_kC_uu - \frac{i}{2}K_u^2P_k(u) - i\alpha(u|1)P_k(1)\end{aligned}$$

thus

$$\frac{du_k'}{dt} = -iT_{|u|^2}u_k' - i\alpha(u \mid 1)\frac{(1 \mid u_k')}{(u_k' \mid u_k')}u_k' \ . \tag{2.4.10}$$

Then (2.4.9) and (2.4.10) obtained above lead to

$$\left(\dot{\Psi}_k - i\left(\sigma_k^2 + 2\alpha\mathrm{Re}\big[\frac{(u \mid 1)(1 \mid u_k')}{(u_k' \mid u_k')}\big]\right)\Psi_k\right)u_k' = -i[T_{|u|^2}, \Psi_k](u_k') \ .$$

We claim that

$$[T_{|u|^2}, \Psi_k](u_k') = 0 \ .$$

therefore

$$\Psi_k(t) = \mathrm{e}^{i(\sigma_k^2 t + \gamma_k(t))}\Psi_k(0) \ ,$$

where

$$\gamma_k(t) = 2\alpha\int_0^t \frac{\mathrm{Re}[(u(t') \mid 1)(1 \mid u_k'(t')]}{|u_k'(t')|^2}\,dt' \ .$$

It remains to prove the claim (one can also refer to [15, Theorem 8] for the proof). We first prove that, for any $\chi_p(z) = \frac{z-p}{1-pz}$ with $|p| < 1$,

$$[T_{|u|^2}, \chi_p]f = 0$$

for any $f \in F_u(\sigma_k)$ such that $\chi_p f \in F_u(\sigma_k)$. For any L^2 function g,

$$[\Pi, \chi_p]g = (1 - |p|^2)H_{1/(1-\bar{p}z)}(h) \ ,$$

where $\overline{(\mathrm{Id} - \Pi)g} = Sh$. Consequently, the range of $[\Pi, \chi_p]$ is one dimensional, directed by $\frac{1}{1-\bar{p}z}$. In particular, $[T_{|u|^2}, \chi_p]f$ is proportional to $\frac{1}{1-\bar{p}z}$. Since

$$\begin{aligned}([T_{|u|^2}, \chi_p]f|1) &= (T_{|u|^2}\chi_p f - \chi_p T_{|u|^2}f|1) \\ &= (\chi_p f|H_u^2(1)) - (\chi_p|1)(H_u^2 f|1) \\ &= (H_u^2(\chi_p f))|1) - (\chi_p|1)(H_u^2 f|1) \\ &= (\chi_p f - (\chi_p|1)f|u)(u|1) \ ,\end{aligned}$$

We used (2.3.6) to gain the last equality. Since $\chi_p f - (\chi_p|1)f \in F_u(\sigma_k)$ is orthogonal to 1, by Proposition 2.2.2, $\chi_p f - (\chi_p|1)f \in E_u(\sigma_k)$, hence $\chi_p f - (\chi_p|1)f \in F_u(\sigma_k)$ is orthogonal to u. This proves that $[T_{|u|^2}, \chi_p]f = 0$. □

Therefore, we have

Corollary 2.4.2.

$$\mathrm{rk}K_{u(t)} = \mathrm{rk}_{\mathrm{d}}K_{u(t)} = \mathrm{rk}K_{u_0}, \ \textit{a.e. } t < \infty.$$

We know that $\Psi_k(t)$ is defined for every t in an open subset Ω of $\mathbb{R}$ consisting of the complement of a discrete closed subset, corresponding to crossings at σ_k^2. Furthermore, by Proposition 2.4.2, on each connected component of Ω, the zeroes of $\Psi_k(t)$ are constant. Together with the following property, $\Psi_k(t)$ never changes it orbit even after the crossings.

Proposition 2.4.3. *For every time t such that $\Psi_k(t)$ is defined, the zeroes of $\Psi_k(t)$ are the same.*

Démonstration. The proposition is a consequence of the following lemma. □

Lemma 2.4.3. *There exists an analytic function $\Psi_k^\sharp$ defined in a neighborhood Ω' of Ω^c and valued into rational functions, and, for every $t \in \Omega \cap \Omega'$, there exists $\beta(t) \in \mathbb{T}$ such that*

$$\Psi_k(t,z) = \mathrm{e}^{i\beta_k(t)}\Psi_k^\sharp(t,z) \ .$$

Démonstration. Since σ_k^2 is an eigenvalue of constant multiplicity m of $K_{u(t)}^2$, the orthogonal projector $P_k(t)$ onto $F_{u(t)}(\sigma_k)$ is an analytic function of $t \in \mathbb{R}$. Consequently, the vector

$$v_k'(t) := P_k(t)(1)$$

depends analytically on t. Furthermore, $v_k'(t)$ is not 0 if $t \notin \Omega$. Indeed, from the description of $F_u(\tau)$ provided by Proposition 2.2.2 when τ is a singular value associated to the pair (H_u, K_u), we observe that, if τ is H dominant, the space $F_u(\tau)$ is not orthogonal to 1. Consequently, we can define, for t in a neighborhood Ω' of Ω^c,

$$\Psi_k^\sharp(t,z) := \frac{K_{u(t)}(v_k'(t))(z)}{\sigma_k v_k'(t,z)}$$

as an analytic function of t valued into rational functions of z. On the other hand, if $t \in \Omega$, Proposition 2.2.2 shows that

$$F_{u(t)}(\sigma_k) \cap u(t)^\perp = E_{u(t)}(\sigma_k) = F_{u(t)}(\sigma_k) \cap 1^\perp \ ,$$

therefore $v_k'(t)$ is collinear to $u_k'(t)$,

$$v_k'(t) = (1|u_k'(t))\frac{u_k'(t)}{\|u_k'(t)\|^2} \ .$$

Since, from the definition of $\Psi_k(t)$,

$$K_{u(t)}(u_k'(t)) = \sigma_k \Psi_k(t) u_k'(t) \ ,$$

we infer that there exists an analytic β_k on $\Omega \cap \Omega'$ valued into $\mathbb{T}$ such that

$$K_{u(t)}(v_k'(t)) = \sigma_k \mathrm{e}^{-i\beta_k(t)}\Psi_k(t) v_k'(t) \ .$$

This completes the proof. □

2.5 Necessary condition of norm explosion

In this section, let $u(t)$ be the solution of α–Szegő equation (2.1.1) with initial data $u_0 \in \mathcal{L}(N)$, $N \in \mathbb{N}^+$, $u^\infty = \lim u(t_n)$ for the weak * topology of $H^{1/2}$, for some sequence t_n going to infinity. To study the large time behavior of solutions, it is equivalent to study the rank of the shifted Hankel operator K_u.

Lemma 2.5.1. *The solution $u(t)$ to the α–Szegő equation will stay in a compact subset of $\mathcal{L}(N)$ if and only if for all the adherent values u^∞ of $u(t)$ at infinity,*

$$\mathrm{rk}K_{u^\infty} = \mathrm{rk}K_{u_0} \ . \tag{2.5.1}$$

Démonstration. By the explicit formula of functions in $\mathcal{L}(N) \subset H^s$ for every s in Theorem 2.1.3, $\mathrm{rk}K_u = N$ if and only if

$$u(z) = \frac{A(z)}{B(z)}$$

with $A, B \in \mathbb{C}_N[z], A \wedge B = 1, \deg(A) = N$ or $\deg(B) = N, B^{-1}(\{0\}) \cap \overline{\mathbb{D}} = \emptyset$.

Then a sequence of $(u_n)_n$ is in a relatively compact subset of $\mathcal{L}(N)$ unless one of the poles of u_n approaches the unit disk $\mathbb{D}$, then the corresponding limit $u(z)$ will be in some $\mathcal{L}(N')$ with $N' < N$. □

We first present a necessary condition of the norm explosion for any $\alpha \in \mathbb{R} \setminus \{0\}$.

Theorem 2.5.1. *If $\mathrm{rk}K_{u^\infty} < \mathrm{rk}K_{u_0}$, then there exists some k such that $\ell_k(u_0) = 0$.*

Corollary 2.5.1. *If $\alpha < 0$, for any $N \in \mathbb{N}^+$, given initial data $u_0 \in \mathcal{L}(N)$, then the solution to the α–Szegő equation stays in a compact subset of $\mathcal{L}(N)$.*

Proof of Corollary 2.5.1. Since $\alpha < 0$, then $\ell_k := \|u_k'\|^2 - \alpha\|v_k'\|^2 > 0$, due to Theorem 2.5.1, $\mathrm{rk}K_{u^\infty} \equiv \mathrm{rk}K_{u_0}$. □

Proof of Theorem 2.5.1. Assume $\mathrm{rk}K_{u^\infty} < \mathrm{rk}K_{u_0}$, then there exists some k such that $\dim F_{u^\infty}(\sigma_k) < \dim F_{u_0}(\sigma_k) = m$. We are to prove $\|u_k'^\infty\|^2 = 0$ and $\|v_k'^\infty\|^2 = 0$.

— $\|u_k'^\infty\|^2 = 0$.

There exists a time dependent Blaschke product Ψ_k of degree $m-1$ such that

$$K^2_{u(t_n)}(u_k'(t_n)) = \sigma_k^2 u_k'(t_n) \ , \ K_{u(t_n)}(u_k'(t_n)) = \sigma_k \Psi_k(t_n) u_k'(t_n) \ , \tag{2.5.2}$$

By Proposition 2.4.3, any limit point of $\Psi_k(t)$ as t goes to ∞ is of degree $m-1$ as well. Since $u_k'(t_n)$ is bounded in L^2_+, up to a subsequence it converges weakly to some $u_k'^\infty \in L^2_+$. Passing to the limit in the identities (2.5.2), we get

$$K^2_{u^\infty}(u_k'^\infty) = \sigma_k^2 u_k'^\infty \ , \ K_{u^\infty}(u_k'^\infty) = \sigma_k \Psi_k^\infty u_k'^\infty \ , \tag{2.5.3}$$

where Ψ_k^∞ is a Blaschke product of degree $m-1$. The latter identities (2.5.3) show that $u_k'^\infty$ and $\Psi_k^\infty u_k'^\infty$ belong to $F_{u^\infty}(\sigma_k)$, hence, if $u_k'^\infty$ is not zero, the dimension of $F_{u^\infty}(\sigma_k)$ is at least m. Indeed, if we write $\Psi_k^\infty = \mathrm{e}^{-i\psi}\frac{P(z)}{D(z)}$, then

$$F_{u^\infty}(\sigma_k) = \left\{ \frac{f}{D(z)} u_k'^\infty, \ f \in \mathbb{C}_{m-1}[z] \right\} \ . \tag{2.5.4}$$

— $\|v_k'^{\infty}\|^2 = 0$.
Recall the structure of $F_u(\sigma_k)$ with $\sigma_k \in \Sigma_K(u)$ in Proposition 2.2.2, the orthogonal projection of 1 onto the space $F_u(\sigma_k)$, v_k' can be represented as

$$v_k' = \left(1 \mid \frac{u_k'}{\|u_k'\|}\right)\frac{u_k'}{\|u_k'\|} \; .$$

If $v_k'^{\infty} \neq 0$, since $\|v_k'\| = \left|(1 \mid \frac{u_k'}{\|u_k'\|})\right|$, thus $\frac{u_k'}{\|u_k'\|} \rightharpoonup v$ in L^2 with $v \neq 0$. Using the strategy in the first step above by replacing u_k' by $\frac{u_k'}{\|u_k'\|}$, we have $\dim F_{u^\infty}(\sigma_k) = m$.
□

2.6 Large time behavior of the solution for the case $\alpha > 0$

In this section, we prove for any N, there exist solutions in $\mathcal{L}(N)$ which admit an exponential on time norm explosion.

Theorem 2.6.1. *For $\alpha > 0$, $u_0 \in H^s_+$ such that $\Sigma_K(u_0) = \{\sigma\}$ with multiplicity $k = \mathrm{rk} K_{u_0}$. Then $\|u(t)\|_{H^s}$ grows exponentially on time,*

$$\|u(t)\|_{H^s} \simeq \mathrm{e}^{C_\alpha(2s-1)|t|} \; ,$$

if and only if

$$L_1(u) := (K_u^2(u)|u) - \alpha(K_u^2(1)|1) = 0 \; . \tag{2.6.1}$$

Let u_0 as in the theorem above. If u_0 is not a Blaschke product, we have

$$\Sigma_H(u_0) = \{\rho_1, \rho_2\} \; , \; \rho_1 > \sigma > \rho_2 \; .$$

Using the results by Gérard and Grellier [15], we have the explicit formula for the solution u as

$$u(t,z) = \frac{\triangle_{11} - \triangle_{21}}{\det(\mathcal{C}(z))}\mathrm{e}^{-i\varphi_1} + \frac{\triangle_{22} - \triangle_{12}}{\det(\mathcal{C}(z))}\mathrm{e}^{-i\varphi_2} \; , \tag{2.6.2}$$

with $\triangle_{jk}$ as the minor determinant of $\mathcal{C}(z)$ corresponding to line k and column j, and

$$\mathcal{C}(z) = \begin{pmatrix} \frac{\rho_1 - \sigma z \Psi \mathrm{e}^{-i\varphi_1}}{\rho_1^2 - \sigma^2} & \frac{\rho_2 - \sigma z \Psi \mathrm{e}^{-i\varphi_2}}{\rho_2^2 - \sigma^2} \\ \frac{1}{\rho_1} & \frac{1}{\rho_2} \end{pmatrix}$$

Then

$$u(t,z) = \frac{(\frac{1}{\rho_2} - \frac{\rho_2 - \sigma z \Psi \mathrm{e}^{-i\varphi_2}}{\rho_2^2 - \sigma^2})\mathrm{e}^{-i\varphi_1} + (\frac{\rho_1 - \sigma z \Psi \mathrm{e}^{-i\varphi_1}}{\rho_1^2 - \sigma^2} - \frac{1}{\rho_1})\mathrm{e}^{-i\varphi_2}}{\frac{1}{\rho_2}(\frac{\rho_1 - \sigma z \Psi \mathrm{e}^{-i\varphi_1}}{\rho_1^2 - \sigma^2}) - \frac{1}{\rho_1}(\frac{\rho_2 - \sigma z \Psi \mathrm{e}^{-i\varphi_2}}{\rho_2^2 - \sigma^2})} \; .$$

An interesting fact is that u is under the form

$$u(t,z) = b(t) + \frac{c'(t) z \Psi(t,z)}{1 - p'(t) z \Psi(t,z)} \; ,$$

where b , p', $c' \in \mathbb{C}$. Since $\Psi(t,z) = \mathrm{e}^{i\psi(t)}\chi(z)$ with χ as a time independent Blaschke product, we then rewrite

$$u(t,z) = b(t) + \frac{c(t)z\chi(z)}{1-p(t)z\chi(z)} \ . \tag{2.6.3}$$

Lemma 2.6.1. *Let χ be a time-independent Blaschke product. A function $u \in C^\infty(\mathbb{R}, H^s_+)$ with $s > \frac{1}{2}$ is a solution of the α–Szegő equation,*

$$i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1) \ ,$$

if and only if

$$\widetilde{u}(t,z) := u(t, z\chi(z))$$

satisfies the α–Szegő equation.

Démonstration. First of all, $z\chi(z) \in C^\infty_+(\mathbb{S}^1)$, then $(z\chi(z))^n \in C^\infty_+(\mathbb{S}^1)$ for any n, so that $u \in H^s_+$ implies $\widetilde{u} \in H^s_+$. Assume u is a solution of the α–Szegő equation, it is equivalent to

$$i\partial_t \hat{u}(t,n) = \sum_{p-q+r=n} \hat{u}(t,p)\overline{\hat{u}}(t,q)\hat{u}(t,r) + \alpha\hat{u}(t,0)\delta_{n0} \ , \ \forall n \geq 0 \ . \tag{2.6.4}$$

Since

$$\Pi(|u(z\chi(z))|^2 u(z\chi(z))) = \sum_{p-q+r\geq 0} \hat{u}(p)\overline{\hat{u}}(q)\hat{u}(r)(z\chi(z))^{p-q+r} \ ,$$

we obtain that $\widetilde{u}$ satisfies the α–Szegő equation.

Conversely, assume $\widetilde{u}$ satisfies the α–Szegő equation, then we have

$$i\partial_t \hat{u}(n)(z\chi(z))^n = \sum_{p-q+r\geq 0} \hat{u}(p)\overline{\hat{u}}(q)\hat{u}(r)(z\chi(z))^{p-q+r} + \hat{u}(0) \ . \tag{2.6.5}$$

Identifying the Fourier coefficients of 0 mode of both sides, we get equation (2.6.4) with $n = 0$. Then withdraw this quantity from both sides of (2.6.5) and simplify by $z\chi(z)$. Continuing this process, we get all the equations (2.6.4) for every n. □

Lemma 2.6.2. *Let Ψ be a Blaschke product of finite degree d and $s \in [0,1)$. There exists $C_{\Psi,s} > 0$ such that, for every $p \in \mathbb{D}$,*

$$\left\| \frac{1}{1-p\Psi} \right\|_{H^s(\mathbb{S}^1)} \geq \frac{C_{\Psi,s}}{(1-|p|)^{s+\frac{1}{2}}} \ .$$

Démonstration. It is a classical fact that, for every $u \in H^s_+(\mathbb{S}^1)$, for every $s \in [0,1)$,

$$\|u\|^2_{H^s(\mathbb{S}^1)} \simeq \int_{\mathbb{D}} |u'(z)|^2(1-|z|^2)^{1-2s}\, dL(z) \ ,$$

where L denotes the bi-dimensional Lebesgue measure.

Let $p \in \mathbb{D}$ close to the unit circle and

$$\omega := \frac{p}{|p|} \ .$$

Since Ψ is a Blaschke product of finite degree d, the equation

$$\omega\Psi(z) = 1$$

admits d solutions on the circle. Moreover, these solutions are simple. Indeed, writing

$$\Psi(z) = \mathrm{e}^{-i\psi} \prod_{j=1}^{d} \frac{z - p_j}{1 - \overline{p}_j z} \ , \ |p_j| < 1 \ ,$$

we have, for every $z \in \mathbb{S}^1$,

$$\frac{\Psi'(z)}{\Psi(z)} = \frac{1}{z} \sum_{j=1} \frac{1 - |p_j|^2}{|z - p_j|^2} \neq 0 \ .$$

Let α be such a solution. For every z such that

$$|z - \alpha| \leq (1 - |p|),$$

we have, if $1 - |p|$ is small enough,

$$|1 - p\Psi(z)| = |1 - p\Psi(\alpha) - p\Psi'(\alpha)(z - \alpha) + O(|z - \alpha|^2)| \leq C(1 - |p|).$$

Therefore

$$\begin{aligned}
\left\| \frac{1}{1 - p\Psi} \right\|^2_{H^s(\mathbb{S}^1)} &\geq A_s \int_{\mathbb{D} \cap \{|z-\alpha| \leq (1-|p|)\}} \left| \frac{\Psi'(z)}{(1 - p\Psi(z))^2} \right|^2 (1 - |z|^2)^{1-2s} \, dL(z) \\
&\geq B_{\Psi,s} (1 - |p|)^{-4} \int_{\mathbb{D} \cap \{|z-\alpha| \leq (1-|p|)\}} (1 - |z|^2)^{1-2s} \, dL(z) \\
&\geq \frac{C^2_{\Psi,s}}{(1 - |p|)^{2s+1}} \ .
\end{aligned}$$

□

Let us turn back to prove the theorem.

Proof of Theorem 2.6.1. Recall that

$$\begin{aligned}
L_1(u) &= \big(K_u^2(u) \mid u\big) - \alpha\big(K_u^2(1) \mid 1\big) \\
&= \frac{1}{2}\big(\|u\|^4_{L^4} - \|u\|^4_{L^2}\big) - \alpha\big(\|u\|^2_{L^2} - |(u \mid 1)|^2\big) \ .
\end{aligned}$$

Since $\chi(z)$ is an inner function, we have

$$(\widetilde{u} \mid \widetilde{v}) = (u|v) \ , \forall \ u, v \ ,$$

thus

$$(\widetilde{u}|1) = (u|1) \ , \|\widetilde{u}\|_{L^2} = \|u\|_{L^2} \ ,$$

and since

$$\widetilde{u^2} = (\widetilde{u})^2 \ ,$$

then

$$\|\widetilde{u}\|_{L^4} = \|u\|_{L^4} \ .$$

As a consequence, $L_1(u) = L_1(\widetilde{u}) = 0$.

The solution $\widetilde{u}$ is under the form (2.6.3),

$$u(t,z) = b(t) + \frac{c(t)z\chi(z)}{1 - p(t)z\chi(z)} = b - \frac{c}{p} + \frac{c}{p}\frac{1}{1 - pz\chi(z)} \ ,$$

thus

$$\begin{aligned} \|u\|_{H^s} &\simeq |c|\|\frac{1}{1 - pz\chi(z)}\|_{H^s} \\ &\geq C_{\chi,s}\frac{|c|}{(1-|p|)^{s+1/2}} \ , \end{aligned}$$

where we used Lemma 2.6.2. Using the result in [50, Theorem 3.1] and its proof, we have

$$\frac{|c|}{(1-|p|)^{s+1/2}} \simeq (1-|p|)^{-s+1/2} \simeq \mathrm{e}^{C_\alpha(2s-1)|t|} \ .$$

Therefore, $\widetilde{u}$ admit an exponential on time growth of the Sobolev norm H^s with $s > \frac{1}{2}$. The proof is complete. $\square$

2.7 Perspectives

The main purpose of this work is to study the dynamics of the general solutions of the α–Szegő equation (2.1.1). We have already observed the weak turbulence by considering some special rational data. We proved the existence of data with exponential in time growth, a natural question is about the genericity of data with such a high growth. Besides, an important open problem is to gain new informations on the solutions with infinite rank.

Another interesting question is about the cubic Szegő equation with other perturbations, for example, consider a Hamiltonian function

$$E(u) = \frac{1}{4}\|u\|_{L^4}^4 + \frac{1}{2}F(|(u|1)|^2) \ ,$$

with a non linear function F. In this case, we still have one Lax pair (K_u, C_u) while the conservation laws we found no longer exist. The question is to study the integrability and also the existence of turbulent solutions of this new Hamiltonian system.

Chapitre 3

Modified scattering theory for a wave guide nonlinear Schrödinger equation

3.1 Introduction

The purpose of this work is to study the large time behavior of solutions to the following Hamiltonian equation. On the cylinder $\mathbb{R}_x \times \mathbb{T}_y$, consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R} \times \mathbb{T})$ with the symplectic form

$$\omega(u, v) = \mathrm{Im}(u|v)$$

and the Hamiltonian function on $\mathcal{H}$,

$$H(U) = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} (|\partial_x U(x,y)|^2 + |D_y| U(x,y) \overline{U}(x,y))\, dx\, dy + \frac{1}{4} \int_{\mathbb{R}\times\mathbb{T}} |U(x,y)|^4 \, dx\, dy \ ,$$

where $|D_y| := \sqrt{-\partial_{yy}}$. The corresponding Hamiltonian system turns out to be a wave guide nonlinear Schrödinger equation,

$$(i\partial_t + \mathcal{A})\, U = |U|^2 U, \ \ (x, y) \in \mathbb{R} \times \mathbb{T} \ , \tag{3.1.1}$$

where we set

$$\mathcal{A} := \partial_{xx} - |D_y| \ .$$

Notice that, besides the energy $H(U)$, this equation formally enjoys the mass conservation law

$$\int_{\mathbb{R}\times\mathbb{T}} |U(t,x,y)|^2 \, dx\, dy = \int_{\mathbb{R}\times\mathbb{T}} |U(0,x,y)|^2 \, dx\, dy \ .$$

In particular, the trajectories are bounded in $H^1_x L^2_y \cap L^2_x H^{\frac{1}{2}}_y$. These conservation laws correspond to a critical regularity for equation (3.1.1), so that global wellposedness of the Cauchy problem is not easy. In this paper, we shall prove that global solutions do exist for every Cauchy datum satisfying a smallness assumption in an appropriate high regularity norm. However, our main objective in this paper is to study the possible large time unboundedness of the solution, in a slightly more regular norm than the energy norm, typically $L^2_x H^s_y(\mathbb{R}\times\mathbb{T})$ for $s>\frac{1}{2}$. This general question of existence of unbounded Sobolev trajectories comes back to [4], and was addressed by several authors for various Hamiltonian PDEs, see e.g. [7, 12, 22, 24, 25, 26, 27, 30, 44, 50]. The choice of the equation (3.1.1) is naturally based on the state of the art for this question concerning the nonlinear Schrödinger equation and the cubic half wave equation, which we recall in the next paragraphs.

3.1.1 Motivation

In this paragraph, we briefly recall the state of the art about the existence of unbounded Sobolev trajectories for the nonlinear Schrödinger equation and the cubic half wave equation.

The nonlinear Schrödinger equation

Firstly, consider the following Schrödinger equation with smooth initial data

$$i\partial_t u + \Delta u = |u|^2 u \ . \tag{3.1.2}$$

If we consider the case with spatial domain $\mathbb{R}$ or $\mathbb{T}$, the $1D$ Schrödinger turns out to be globally well-posed and completely integrable [52], and the higher conservation laws in that case imply

$$\|u(t)\|_{H^s} \le C_s(\|u(0)\|_{H^s}) \ , \ \ s\ge 1 \ , \ \text{ for all } t\in\mathbb{R} \ .$$

Hani–Pausader–Tzvetkov–Visciglia studied the nonlinear Schrödinger on the cylinder $\mathbb{R}_x\times\mathbb{T}^d_y$ [26], they found infinite cascade solutions for $d\ge 2$, which means there exists solutions with small Sobolev norms at the initial time, while admit infinite Sobolev norms when time goes to infinity.

Theorem 3.1.1. *[26, Corollary 1.4] Let $d\ge 2$ and $s\in\mathbb{N}$, $s\ge 30$. Then for every $\varepsilon>0$ there exists a global solution $U(t)$ of the cubic Schrödinger equation (3.1.2) on $\mathbb{R}\times\mathbb{T}^d$, such that*

$$\|U(0)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)}\le\varepsilon, \qquad \limsup_{t\to+\infty}\|U(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^d)}=+\infty. \tag{3.1.3}$$

Unfortunately, these infinite cascades do not occur for $d=1$, actually the dynamics of small solutions is fairly similar on $\mathbb{R}\times\mathbb{T}$ and $\mathbb{R}$. But we may apply their general strategy to the wave guide Schrödinger equation, to understand the asymptotic behavior and in particular how this asymptotic behavior is related to *resonant dynamics*.

The half wave equation

Another motivation is from the study of the so-called *half wave* equation [12]. Actually, if we start with a solution u which does not depend on x, then it satisfies the following half wave equation

$$i\partial_t u - |D_y| u = |u|^2 u, \ y\in\mathbb{T} \ . \tag{3.1.4}$$

The following theorem was proved by Gérard and Grellier, which tells us the global well-posedness and partially about its large time behavior. The orthogonal projector from $L^2(\mathbb{T})$ onto

$$L^2_+(\mathbb{T}) := \left\{ u(y) = \sum_{p\geq 0} u_p \mathrm{e}^{ipy},\ (u_p)_{p\geq 0} \in \ell^2 \right\} ,$$

is called the Szegő projector and is denoted by Π_+.

Theorem 3.1.2. *[12] Given $u_0 \in H^{\frac{1}{2}}(\mathbb{T})$, there exists a unique solution $u \in C(\mathbb{R}, H^{\frac{1}{2}}(\mathbb{T}))$ satisfying (3.1.4). And if $u_0 \in H^s(\mathbb{T})$ for some $s > \frac{1}{2}$, then $u \in C(\mathbb{R}, H^s(\mathbb{T}))$. Moreover, let $s > 1$ and $u_0 = \Pi_+(u_0) \in L^2_+(\mathbb{T}) \cap H^s(\mathbb{T})$ with $\|u_0\|_{H^s} = \varepsilon$, $\varepsilon > 0$ small enough. Denote by v the solution of the cubic Szegő equation[11, 13]*

$$i\partial_t v - Dv = \Pi_+(|v|^2 v) \ ,\ v(0,\cdot) = u_0 \ . \tag{3.1.5}$$

Then, for any $\alpha > 0$, there exists a constant $c = c_\alpha < 1$ so that

$$\|u(t) - v(t)\|_{H^s} = \mathcal{O}(\varepsilon^{3-\alpha}) \text{ for } t \leq \frac{c_\alpha}{\varepsilon^2} \log\frac{1}{\varepsilon} \ . \tag{3.1.6}$$

A similar result is available for the case on the real line $\mathbb{R}$, see O. Pocovnicu [46].

The following large time behavior result of the half wave equation comes from the fact that the cubic Szegő dynamics which appears as the effective dynamics, admits large time Sobolev norm growth.

Corollary 3.1.1. *[12] Let $s > 1$. There exists a sequence of data u_0^n and a sequence of times t^n such that, for any r,*

$$\|u_0^n\|_{H^r} \to 0$$

while the corresponding solution of (3.1.4) satisfies

$$\|u^n(t^n)\|_{H^s} \simeq \|u_0^n\|_{H^s} \left(\log \frac{1}{\|u_0^n\|_{H^s}}\right)^{2s-1} \ .$$

Remark 3.1.1. *In the statement above, one may observe that there exists norm growth, but $\|u^n(t^n)\|_{H^s}$ stays still small. In fact, it is possible to show that for $s > 1$, there exists a sequence of u^n solutions to the half wave equation (3.1.4) such that [42]*

$$\|u_0^n\|_{H^s} \to 0,\ \|u^n(t^n)\|_{H^s} \to \infty \ . \tag{3.1.7}$$

Indeed, one may just take some large integers $N_n = \left[\left(\|\widetilde{u_0^n}\|_{H^s}\|\widetilde{u^n}(t^n)\|_{H^s}\right)^{-\frac{1}{1+2s}}\right]$, set $u_0^n = N_n^{\frac{1}{2}}\widetilde{u_0^n}(N_n^{\frac{1}{2}}y)$ with $\widetilde{u_0^n}$ given as in Corollary 3.1.1, then we may write the related solution as $u_0^n = N_n^{\frac{1}{2}}\widetilde{u^n}(N_n t, N_n^{\frac{1}{2}}y)$.

The existence of a solution to the half wave equation (3.1.4) satisfying

$$\|u_0\|_{H^s} \leq \varepsilon, \quad \limsup_{t\to\infty} \|u(t)\|_{H^s} = \infty \ , \tag{3.1.8}$$

is still an open problem. Though this problem is still open for the half wave equation, we are going to solve it for the wave guide Schrödinger equation (3.1.1).

3.1.2 Main results

The aim of this paper is to describe the large time behavior of the wave guide Schrödinger equation (3.1.1) for small smooth data. Thoughout this paper, we always assume the initial data satisfy

$$U_0(x, y+\pi) = -U_0(x,y)\ . \tag{3.1.9}$$

A direct consequence is that U_0 only admits odd Fourier modes on the direction y, which is of helpful importance in the study of the resonant system, as we will see later in section 4. We then show that the asymptotic dynamics of small solutions to (3.1.1) is related to that of solutions of the resonant system

$$\begin{aligned} &i\partial_t G_\pm(t) = \mathcal{R}[G_\pm(t), G_\pm(t), G_\pm(t)]\ , \\ &\mathcal{F}_{\mathbb{R}}\mathcal{R}[G_\pm, G_\pm, G_\pm](\xi, y) = \Pi_\pm(|\widehat{G}_\pm|^2\widehat{G}_\pm)(\xi, y)\ . \end{aligned} \tag{3.1.10}$$

Here $\widehat{G}(\xi,\cdot) = \mathcal{F}_{\mathbb{R}}G(\xi,\cdot)$, Π_+ is the Szegő projector onto the non-negative Fourier modes, $\Pi_- := \mathrm{Id} - \Pi_+$, and $G_\pm := \Pi_\pm(G)$. Noting that the dependence on ξ is merely parametric, the above system is none other than the resonant system for the cubic half wave equation on $\mathbb{T}$, which is the cubic Szegő equation.

Throughout this article, we assume $N \geq 13$ is an arbitrary integer, and $\delta < 10^{-3}$. Our main results on the modified scattering and the existence of a wave operator are as below, where the norms of Banach spaces S and S^+ are defined as

$$\|F\|_S := \|F\|_{H^N_{x,y}} + \|xF\|_{L^2_{x,y}}, \quad \|F\|_{S^+} := \|F\|_S + \|(1-\partial_{xx})^4 F\|_S + \|xF\|_S. \tag{3.1.11}$$

Theorem 3.1.3. *There exists $\varepsilon = \varepsilon(N) > 0$ such that if $U_0 \in S^+$ satisfies*

$$\|U_0\|_{S^+} \leq \varepsilon,$$

and if $U(t)$ solves (3.1.1) with initial data U_0, then $U \in C([0,+\infty) : S)$ exists globally and exhibits modified scattering to its resonant dynamics (3.1.10) in the following sense : there exists $G_0 \in S$ such that if $G(t)$ is the solution of (3.1.10) with initial data $G(0) = G_0$, then

$$\|U(t) - \mathrm{e}^{it\mathcal{A}}G(\pi \ln t)\|_S \to 0 \quad \text{as } t \to +\infty.$$

Remark 3.1.2. *The Cauchy problem of our wave guide Schrödinger system (3.1.1) in the classical Sobolev space is not easy, neither by energy estimates nor by Strichartz estimates, since its Hamiltonian energy lies on the Sobolev space $H^1_xL^2_y \cap L^2_xH^{1/2}_y$. However, by the Theorem 3.1.3 above, we can deduce directly the global well-posedness with small initial data in S^+.*

Theorem 3.1.4. *There exists $\varepsilon = \varepsilon(N) > 0$ such that if $G_0 \in S^+$ satisfies*

$$\|G_0\|_{S^+} \leq \varepsilon,$$

$G(t)$ solves (3.1.10) with initial data G_0, then there exists $U \in C([0,\infty) : S)$ a solution of (3.1.1) such that

$$\|U(t) - \mathrm{e}^{it\mathcal{A}}G(\pi \ln t)\|_S \to 0 \quad \text{as } t \to +\infty.$$

Theorem 3.1.4 combined with the large time behavior of the cubic Szegő equation, leads to the infinite cascades result.

Theorem 3.1.5. *Given $N \geq 13$, then for any $\varepsilon > 0$, there exists $U_0 \in S^+$ with $\|U_0\|_{S^+} \leq \varepsilon$, such that the corresponding solution to (3.1.1) satisfies*

$$\limsup_{t\to\infty} \|U(t)\|_{L_x^2 H_y^s} = \infty \ , \quad \forall s > 1/2 \ . \tag{3.1.12}$$

Remark 3.1.3.
1. *It is likely there exists a dense G_δ set in an appropriate space containing initial data which lead to infinite cascade as above. A proof of this would involve more technicalities and we will not discuss it in this paper.*
2. *Compared to the results in [26], the unbounded Sobolev norms in our theorem are just above the energy norm.*

3.1.3 Organization of this chapter

In section 2, we introduce the notation used in this chapter. In section 3, we study the structure of the non-linearity, and establish the decomposition proposition, which is of crucial importance. We decompose the non-linearity $\mathcal{N}^t$ into a combination of the resonant part and a remainder,

$$\mathcal{N}^t[F,G,H] = \frac{\pi}{t}\mathcal{R}[F,G,H] + \mathcal{E}^t[F,G,H] \ .$$

In section 4, we study the resonant system and its large time cascade, which is similar to the cubic Szegő equation as above. In section 5, we construct the modified wave operator and prove Theorem 3.1.4 and Theorem 3.1.3. Later in this section, we prove the large time blow up result, Theorem 3.1.5. Finally in section 6, we present a lemma that will allow us to transfer L^2 estimates on operators into estimates in S and S^+ norms.

3.2 Preliminary

3.2.1 Notation

We will follow the notation of [26], $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, the inner product $(U,V) := \int_{\mathbb{R}\times\mathbb{T}} U\overline{V}dxdy$ for any $U,V \in L^2(\mathbb{R}\times\mathbb{T})$. We will use the lower-case letter to denote functions $f : \mathbb{R} \to \mathbb{C}$ and the capital letters to denote functions $F : \mathbb{R}\times\mathbb{T} \to \mathbb{C}$, and calligraphic letters denote operators, except for the Littlewood-Paley operators and dyadic numbers which are capitalized most of the time.

We use a different notation to denote Fourier transform on different space variables. The Fourier transform on $\mathbb{R}$ is defined by

$$\widehat{g}(\xi) := \mathcal{F}_x(g)(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-ix\xi} g(x)dx \ .$$

Similarly, if $U(x,y)$ depends on $(x,y) \in \mathbb{R}\times\mathbb{T}$, $\widehat{U}(\xi,y)$ denotes the partial Fourier transform in x. The Fourier transform of $h:\mathbb{T}\to\mathbb{C}$ is,

$$h_p := \mathcal{F}_y(h)(p) = \frac{1}{2\pi}\int_{\mathbb{T}} h(y)\mathrm{e}^{-ipy}dy, \qquad p\in\mathbb{Z}\ ,$$

and this also extends to $U(x,y)$. Finally, we define the full Fourier transform on the cylinder $\mathbb{R}\times\mathbb{T}$

$$\left(\mathcal{F}U\right)(\xi,p) = \frac{1}{2\pi}\int_{\mathbb{T}} \widehat{U}(\xi,y)\mathrm{e}^{-ipy}dy = \widehat{U}_p(\xi)\ .$$

We will often use Littlewood-Paley projections. For the full frequency space, these are defined as follows with N as a dyadic integer.

$$\left(\mathcal{F}P_{\leq N}U\right)(\xi,p) = \varphi(\frac{\xi}{N})\varphi(\frac{p}{N})\left(\mathcal{F}U\right)(\xi,p)\ ,$$

where $\varphi\in C_c^\infty(\mathbb{R})$, $\varphi(x)=1$ when $|x|\leq 1$ and $\varphi(x)=0$ when $|x|\geq 2$. We then also define

$$\phi(x) = \varphi(x)-\varphi(2x) \tag{3.2.1}$$

and

$$P_N = P_{\leq N} - P_{\leq N/2}, \quad P_{\geq N} = 1 - P_{\leq N}\ . \tag{3.2.2}$$

Sometimes we concentrate on the frequency in x only, and we therefore define

$$\left(\mathcal{F}Q_{\leq N}U\right)(\xi,p) = \varphi(\frac{\xi}{N})\left(\mathcal{F}U\right)(\xi,p)\ ,$$

and define Q_N similarly. By a slight abuse of notation, we will consider Q_N indifferently as an operator on functions defined on $\mathbb{R}\times\mathbb{T}$ and on $\mathbb{R}$. While we consider the frequency in y we will use notation Δ_N which means

$$\left(\mathcal{F}_y\Delta_N h\right)(p) = \phi(\frac{p}{N})h_p\ . \tag{3.2.3}$$

We shall use the following commutator estimate which is a direct consequence of the definition,

$$\|[Q_N,x]\|_{L^2_x\to L^2_x} \lesssim N^{-1}\ . \tag{3.2.4}$$

We will use the following sets corresponding to momentum and resonance level sets :

$$\mathcal{M} := \{(p_0,p_1,p_2,p_3)\in\mathbb{Z}^4:\ p_0-p_1+p_2-p_3=0\}\ ,$$

$$\Gamma_\omega := \{(p_0,p_1,p_2,p_3)\in\mathbb{Z}^4:\ |p_0|-|p_1|+|p_2|-|p_3|=\omega\}\ .$$

3.2.2 The non-linearity

Let us write a solution of (3.1.1) as

$$U(x,y,t) = \sum_{p\in\mathbb{Z}} \mathrm{e}^{ipy}e^{-it|p|}\left(\mathrm{e}^{it\partial_{xx}}F_p(t)\right)(x) := \mathrm{e}^{it\mathcal{A}}(F(t))\ ,$$

with $\mathcal{A} = \partial_{xx} - |D_y|$. We then see that U solves (3.1.1) if and only if F solves

$$i\partial_t F(t) = \mathrm{e}^{-it\mathcal{A}}\Big(\mathrm{e}^{it\mathcal{A}}F(t) \cdot \mathrm{e}^{-it\mathcal{A}}\overline{F(t)} \cdot \mathrm{e}^{it\mathcal{A}}F(t)\Big) \ . \tag{3.2.5}$$

We denote the non-linearity in (3.2.5) by $\mathcal{N}^t[F(t), F(t), F(t)]$, where the trilinear form $\mathcal{N}^t$ is defined by

$$\mathcal{N}^t[F, G, H] := \mathrm{e}^{-it\mathcal{A}}\Big(\mathrm{e}^{it\mathcal{A}}F \cdot \mathrm{e}^{-it\mathcal{A}}\overline{G} \cdot \mathrm{e}^{it\mathcal{A}}H\Big) \ .$$

Now, we can compute the Fourier transform of the last expression

$$\mathcal{F}\mathcal{N}^t[F, G, H](\xi, p) = \sum_{(p,q,r,s)\in\mathcal{M}} \mathrm{e}^{it(|p|-|q|+|r|-|s|)}\mathcal{F}_x\big(\mathcal{I}^t[F_q, G_r, H_s]\big)(\xi) \ , \tag{3.2.6}$$

where

$$\mathcal{I}^t[f, g, h] := \mathcal{U}(-t)\Big(\mathcal{U}(t)f\,\overline{\mathcal{U}(t)g}\,\mathcal{U}(t)h\Big), \ \mathcal{U}(t) = \mathrm{e}^{it\partial_{xx}} \ . \tag{3.2.7}$$

One verifies that

$$\mathcal{F}_x\big(\mathcal{I}^t[f, g, h]\big)(\xi) = \int_{\mathbb{R}^2} \mathrm{e}^{it2\eta\kappa}\widehat{f}(\xi-\eta)\overline{\widehat{g}}(\xi-\eta-\kappa)\widehat{h}(\xi-\kappa)d\kappa d\eta \ .$$

3.2.3 Norms

The following Sobolev norms will be used in the whole paper. For sequences $a := \{a_p : \ p \in \mathbb{Z}\}$, we define the following norm,

$$\|a\|^2_{h^s_p} := \sum_{p\in\mathbb{Z}} \left[1 + |p|^2\right]^s |a_p|^2 \ .$$

The Besov space $B^1 = B^1_{1,1}(\mathbb{T})$ is defined as the set of functions f such that $\|f\|_{B^1_{1,1}}$ is finite, where

$$\|f\|_{B^1_{1,1}} = \|S_0(f)\|_{L^1} + \sum_{N \text{ dyadic}} N\|\Delta_N f\|_{L^1},$$

here $f = S_0(f) + \sum_{N \text{ dyadic}} \Delta_N f$ stands for the Littlewood-Paley decomposition of f with Δ_N defined as (3.2.3) above and $\mathcal{F}_y(S_0 f)(p) := \varphi(p)f_p$. The space B^1 will be crucial in the analysis of the resonant system in section 4.

For functions F defined on $\mathbb{R}\times\mathbb{T}$, we will indicate the domain of integration by a subscript x (for $\mathbb{R}$), x, y (for $\mathbb{R}\times\mathbb{T}$) or p (for $\mathbb{Z}$). We will use mainly four different norms : two weak norms

$$\|F\|^2_{Y^s} := \sup_{\xi\in\mathbb{R}} \left[1 + |\xi|^2\right]^2 \sum_p (1 + |p|^2)^s |\widehat{F}_p(\xi)|^2 \ , \tag{3.2.8}$$

$$\|F\|_Z := \sup_{\xi\in\mathbb{R}} \left[1 + |\xi|^2\right] \|\widehat{F}(\xi, \cdot)\|_{B^1} \ , \tag{3.2.9}$$

and two strong norms

$$\|F\|_S := \|F\|_{H^N_{x,y}} + \|xF\|_{L^2_{x,y}} \ , \|F\|_{S^+} := \|F\|_S + \|(1-\partial_{xx})^4 F\|_S + \|xF\|_S \ , \tag{3.2.10}$$

with N to be fixed later.

The space-time norms we will use are

$$\begin{aligned}\|F\|_{X_T} &:= \sup_{0\leq t\leq T} \left\{\|F(t)\|_Z + (1+|t|)^{-\delta}\|F(t)\|_S + (1+|t|)^{1-3\delta}\|\partial_t F(t)\|_S\right\} ,\\ \|F\|_{X_T^+} &:= \|F\|_{X_T} + \sup_{0\leq t\leq T} \left\{(1+|t|)^{-5\delta}\|F(t)\|_{S^+} + (1+|t|)^{1-7\delta}\|\partial_t F(t)\|_{S^+}\right\} ,\end{aligned} \tag{3.2.11}$$

with a small parameter $\delta < 10^{-3}$.

In the following sections, we will see that the Z norm is a conserved quantity for the resonant system, which is of crucial importance, and for data in S^+, the solution is expected to grow slowly in S^+, while the difference between the true solution to (3.1.1) and the solution to the resonant system may decay in S.

Now, at this stage, we present some elementary lemmas which will be useful in the later studies.

Lemma 3.2.1. *Provided $N \geq 13$, we have the following hierarchy*

$$\|F\|_{Y^{1/2}} \lesssim \|F\|_Z \lesssim \|F\|_{Y^s},\ \ s > 1 \ , \tag{3.2.12}$$

$$\|F\|_{H^{1/2}_{x,y}} \lesssim \|F\|_Z \lesssim \|F\|_S \lesssim \|F\|_{S^+} \ . \tag{3.2.13}$$

Démonstration. We begin with the proof of the first inequality (3.2.12), it is sufficient to prove

$$\|f\|_{H^{1/2}_y} \lesssim \|f\|_{B^1} \lesssim \|f\|_{H^s_y},\ \ s > 1 \ .$$

1. $\|f\|_{H^{1/2}_y} \lesssim \|f\|_{B^1}$.

$$\|f\|_{H^{1/2}_y} = \|S_0 f\|_{L^2} + \Big(\sum_{N \text{ dyadic}} N\|\Delta_N f\|^2_{L^2}\Big)^{1/2} \ .$$

We notice that the Fourier transform of $S_0 f$ is compactly supported on some interval I with $|I| < 2$, thus

$$\|S_0 f\|_{L^2} \leq \|\mathcal{F}_y(S_0 f)(p)\|_{\ell^2_p} \leq \left(\sum_{p\in I} |\int \mathrm{e}^{-ixp}(S_0 f)(x)dx|^2\right)^{1/2} \lesssim \|S_0 f\|_{L^1} \ .$$

While the Fourier transform of $\Delta_N f$ is compactly supported on some interval I with $|J_N| \sim N$, thus similarly

$$N^{1/2}\|\Delta_N f\|_{L^2} \leq N^{1/2}\|\mathcal{F}_y(\Delta_N f)(p)\|_{\ell^2_p(J_N)} \lesssim N\|\Delta_N f\|_{L^1} \ ,$$

we then use the fact that ℓ^1 is continuously embedded in ℓ^2 and get

$$\|N^{1/2}\|\Delta_N f\|_{L^2}\|_{\ell^2_N} \leq \|N\|\Delta_N f\|_{L^1}\|_{\ell^2_N} \lesssim \sum_{N \text{ dyadic}} N\|\Delta_N f\|_{L^1} \ ,$$

thus $\|f\|_{H^{1/2}_y} \lesssim \|f\|_{B^1}$.

2. $\|f\|_{B^1} \lesssim \|f\|_{H^s_y}, \ s>1.$
Since $\mathbb{T}$ is of finite measure,

$$\|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T})} \ .$$

This inequality is deduced by Cauchy-Schwarz inequality, indeed,

$$\sum N\|\Delta_N f\|_{L^1} \leq \sum N\|\Delta_N f\|_{L^2} \leq \big(\sum N^{2s}\|\Delta_N f\|^2_{L^2}\big)^{1/2}\big(\sum_{N \text{ dyadic}} N^{-2(s-1)}\big)^{1/2} \ ,$$

the second factor on the right hand side converges since $s>1$, and we obtain our result.
3. It is easy to show the first and last inequality in (3.2.13), and the middle inequality comes from the following Gagliardo-Nirenberg type inequality

$$\|F\|_{Y^s} \lesssim \|F\|^{1/2-\sigma}_{L^2_{x,y}}\|F\|^{1/2+\sigma}_S \ , \ s>1 \ , \tag{3.2.14}$$

with $0<\sigma<1/2$ and the index in the definition of S norm satisfies $\sigma N>3$.

To verify this inequality, we need the elementary inequality

$$\|\widehat{f}\|_{L^\infty_\xi(\mathbb{R})} \lesssim \|f\|_{L^1_x(\mathbb{R})} \lesssim \|f\|^{\frac{1}{2}}_{L^2_x(\mathbb{R})}\|xf\|^{\frac{1}{2}}_{L^2_x(\mathbb{R})}, \tag{3.2.15}$$

one might observe that

$$\begin{aligned}
\big[1+|\xi|^2\big]\,|\widehat{F}_p(\xi)| &\lesssim \sum_{N \text{ dyadic}} N^2|\widehat{Q_N F_p}(\xi)| \\
&\lesssim \sum_N N^2\|Q_N F_p(\cdot)\|^{\frac{1}{2}}_{L^2_x}\|xQ_N F_p(\cdot)\|^{\frac{1}{2}}_{L^2_x} \\
&\lesssim \big(\sum_N N^{-\frac{\theta-4}{2}}\big)\|(1-\partial_{xx})^{\frac{\theta}{2}}F_p(\cdot)\|^{\frac{1}{2}}_{L^2_x}\|\langle x\rangle F_p(\cdot)\|^{\frac{1}{2}}_{L^2_x} \\
&\lesssim \|F_p(\cdot)\|^{\frac{1}{2}}_{H^\theta_x}\|\langle x\rangle F_p(\cdot)\|^{\frac{1}{2}}_{L^2_x} \ ,
\end{aligned}$$

where we applied (3.2.4) to gain the third inequality, and $\theta>4$. Squaring and multiplying by $\langle p\rangle^{2s}$, and combining with (3.2.12), we have for $s>1$,

$$\|F\|^2_{Y^s} = \sup_\xi[1+|\xi|^2]^2\|\widehat{F}(\xi,\cdot)\|^2_{H^s_y} \lesssim \sum_{p\in\mathbb{Z}}\langle p\rangle^{2s}\|F_p(\cdot)\|_{H^\theta_x}\|\langle x\rangle F_p(\cdot)\|_{L^2_x} \lesssim \|F\|_{H^\theta_x H^{2s}_y}\|xF\|_{L^2_{x,y}} \ ,$$

the last inequality comes from the Cauchy-Schwarz inequality. Then (3.2.14) comes from an application of the Gagliardo-Nirenberg inequality on $\|F\|_{H^{\theta+2s}_{x,y}}$ with $\theta+2s>6$,

$$\|F\|_{H^{\theta+2s}_{x,y}} \leq \|F\|^{1-2\sigma}_{L^2_{x,y}}\|F\|^{2\sigma}_{H^N_{x,y}} \ ,$$

and $2\sigma N=\theta+2s>6$. By choosing $\sigma=1/4$ and $N>12$, thus for $s>1$,

$$\|F\|_Z \lesssim \|F\|_{Y^s} \lesssim \|F\|^{\frac{1}{4}}_{L^2_{x,y}}\|F\|^{\frac{3}{4}}_S \ . \tag{3.2.16}$$

□

We remark that by taking suitable σ, for the inequality (3.2.13), the requirement of the Sobolev regularity in S norm may be $N \geq 7$.

We also remark that the operators $Q_{\leq N}$, $P_{\leq N}$ and the multiplication by $\varphi(\cdot/N)$ are bounded in Z, S, S^+, uniformly in N.

In this paper, we make often use of the following elementary bound to sum-up the $1d$ estimates,

$$\Big\| \sum_{q-r+s=p} c_q^1 c_r^2 c_s^3 \Big\|_{\ell_p^2} \lesssim \min_{\{j,k,\ell\}=\{1,2,3\}} \|c^j\|_{\ell_p^2} \|c^k\|_{\ell_p^1} \|c^\ell\|_{\ell_p^1} \ . \tag{3.2.17}$$

The following lemma shows the bounds on the non-linearity $\mathcal{N}^t$ in the S and S^+ norms.

Lemma 3.2.2. *[26, Lemma 2.1]*

$$\begin{aligned} &\|\mathcal{N}^t[F,G,H]\|_S \lesssim (1+|t|)^{-1} \|F\|_S \|G\|_S \|H\|_S \ , \\ &\|\mathcal{N}^t[F^1,F^2,F^3]\|_{S^+} \lesssim (1+|t|)^{-1} \max_{\{j,k,\ell\}=\{1,2,3\}} \|F^j\|_{S^+} \|F^k\|_S \|F^\ell\|_S \ . \end{aligned} \tag{3.2.18}$$

Démonstration. Due to Lemma 3.6.2 in the appendix, it is sufficient to prove

$$\|\mathcal{N}^t[F^1,F^2,F^3]\|_{L^2_{x,y}} \lesssim (1+|t|)^{-1} \min_{\{j,k,\ell\}=\{1,2,3\}} \|F^j\|_{L^2_{x,y}} \|F^k\|_S \|F^\ell\|_S \ . \tag{3.2.19}$$

Coming back to (3.2.6),

$$\|\mathcal{N}^t[F^1,F^2,F^3]\|_{L^2_{x,y}} \lesssim \| \sum_{q-r+s=p} \|\mathcal{I}^t[F_q^1,F_r^2,F_s^3]\|_{L^2_x} \|_{\ell_p^2} \ , \tag{3.2.20}$$

thus we only need to calculate $\|\mathcal{I}^t[f^1,f^2,f^3]\|_{L^2_x}$. By the definition of $\mathcal{I}^t$ (3.2.7), we have the energy bound

$$\begin{aligned} \|\mathcal{I}^t[f^1,f^2,f^3]\|_{L^2_x} &= \Big\| \mathrm{e}^{-it\partial_{xx}} \Big(\mathrm{e}^{it\partial_{xx}} f^1 \overline{\mathrm{e}^{it\partial_{xx}} f^2} \mathrm{e}^{it\partial_{xx}} f^3 \Big) \Big\|_{L^2_x} \\ &\lesssim \min_{\{j,k,\ell\}=\{1,2,3\}} \|f^j\|_{L^2_x} \|\mathrm{e}^{it\partial_{xx}} f^k\|_{L^\infty_x} \|\mathrm{e}^{it\partial_{xx}} f^\ell\|_{L^\infty_x} \ . \end{aligned}$$

Then by (3.2.17),

$$\|\mathcal{N}^t[F^1,F^2,F^3]\|_{L^2_{x,y}} \lesssim \min_{\{j,k,\ell\}=\{1,2,3\}} \|F^j\|_{L^2_{x,y}} \sum_r \|\mathrm{e}^{it\partial_{xx}} F_r^k\|_{L^\infty_x} \sum_s \|\mathrm{e}^{it\partial_{xx}} F_s^\ell\|_{L^\infty_x} \ . \tag{3.2.21}$$

For $|t| > 1$, the factor $(1+|t|)^{-1}$ comes from the dispersive estimate

$$\|\mathrm{e}^{it\partial_{xx}} f\|_{L^\infty_x} \lesssim |t|^{-\frac{1}{2}} \|f\|_{L^1_x} \lesssim |t|^{-\frac{1}{2}} \|f\|_{L^2_x}^{\frac{1}{2}} \|xf\|_{L^2_x}^{\frac{1}{2}} \ , \tag{3.2.22}$$

then

$$\begin{aligned}\sum_p \|\mathrm{e}^{it\partial_{xx}} F_p\|_{L_x^\infty} &\lesssim |t|^{-1/2} \sum_p \|F_p\|_{L_x^2}^{\frac{1}{2}} \|xF_p\|_{L_x^2}^{\frac{1}{2}} \\ &= |t|^{-1/2} \sum_p |p|^{-s}|p|^s \|F_p\|_{L_x^2}^{\frac{1}{2}} \|xF_p\|_{L_x^2}^{\frac{1}{2}} \\ &\le |t|^{-1/2} (\sum_p |p|^{-2s})^{1/2} (\sum_p |p|^{4s} \|F_p\|_{L_x^2}^2)^{1/4} (\sum_p \|xF_p\|_{L_x^2}^2)^{1/4} \\ &\le |t|^{-1/2} \|F\|_S \ ,\end{aligned}$$

where we took $s > 1/2$ in the second and third inequalities. While for $|t| \le 1$, one may use Sobolev estimate instead of the dispersive estimate,

$$\|\mathrm{e}^{it\partial_{xx}} f\|_{L_x^\infty} \lesssim \|f\|_{H_x^1} \ ,$$

then

$$\begin{aligned}\sum_p \|\mathrm{e}^{it\partial_{xx}} F_p\|_{L_x^\infty} \lesssim \sum_p \|F_p\|_{H_x^1} &= \sum_p |p|^{-s}|p|^s \|F_p\|_{H_x^1} \\ &\le (\sum_p |p|^{-2s})^{1/2} (\sum_p |p|^{2s} \|F_p\|_{H_x^1}^2)^{1/2} \le \|F\|_S \ ,\end{aligned}$$

with $s > 1/2$. Thus for any t,

$$\sum_{p\in\mathbb{Z}} \|\mathrm{e}^{it\partial_{xx}} F_p\|_{L_x^\infty} \lesssim (1+|t|)^{-1/2} \|F\|_S \ . \tag{3.2.23}$$

Plugging (3.2.23) into (3.2.21), we get (3.2.19) and complete the proof of Lemma 3.2.2. □

3.3 Structure of the non-linearity

The purpose of this section is to extract the key effective interactions from the full non-linearity in (3.1.1). We are to gain the decomposition

$$\mathcal{N}^t[F,G,H] = \frac{\pi}{t} \mathcal{R}[F,G,H] + \mathcal{E}^t[F,G,H] \ , \tag{3.3.1}$$

where $\mathcal{R}$ is the resonant part,

$$\mathcal{F}\mathcal{R}[F,G,H](\xi,p) = \sum_{(p,q,r,s)\in\Gamma_0} \widehat{F}_q(\xi) \overline{\widehat{G}_r(\xi)} \widehat{H}_s(\xi) \ , \tag{3.3.2}$$

and $\mathcal{E}^t$ is a remainder term, which is estimated in Proposition 3.3.1 below. We will see later that this $\mathcal{R}[G,G,G]$ is exactly the same one as in (3.1.10).

Our main result in this section is the following proposition.

Proposition 3.3.1. *Assume that for $T^* \geq 1$, F, G, $H : \mathbb{R} \to S$ satisfy*

$$\|F\|_{X_{T^*}} + \|G\|_{X_{T^*}} + \|H\|_{X_{T^*}} \leq 1 \ . \tag{3.3.3}$$

Then we can write

$$\mathcal{E}^t[F(t), G(t), H(t)] = \mathcal{E}_1^t[F(t), G(t), H(t)] + \mathcal{E}_2^t[F(t), G(t), H(t)] \ ,$$

and if for $j = 1, 2$ we note $\mathcal{E}_j(t) := \mathcal{E}_j^t[F(t), G(t), H(t)]$ then the following estimates hold uniformly in $T^* \geq 1$,

$$\begin{aligned}
&\sup_{1 \leq T \leq T^*} T^{-\delta} \| \int_{T/2}^{T} \mathcal{E}_j(t) dt \|_S \lesssim 1, \quad j = 1, 2 \ , \\
&\sup_{1 \leq t \leq T^*} (1 + |t|)^{1+\delta} \|\mathcal{E}_1(t)\|_Z \lesssim \sup_{1 \leq t \leq T^*} (1 + |t|)^{1+\delta} \|\mathcal{E}_1(t)\|_{Y^s} \lesssim 1 \ , \ s > 1 \ , \\
&\sup_{1 \leq t \leq T^*} (1 + |t|)^{1/10} \|\mathcal{E}_3(t)\|_S \lesssim 1 \ ,
\end{aligned}$$

where $\mathcal{E}_2(t) = \partial_t \mathcal{E}_3(t)$. Assuming in addition

$$\|F\|_{X^+_{T^*}} + \|G\|_{X^+_{T^*}} + \|H\|_{X^+_{T^*}} \leq 1 \ , \tag{3.3.4}$$

we also have that

$$\sup_{1 \leq T \leq T^*} T^{-5\delta} \| \int_{T/2}^{T} \mathcal{E}_j(t) dt \|_{S^+} \lesssim 1, \ \sup_{1 \leq T \leq T^*} T^{2\delta} \| \int_{T/2}^{T} \mathcal{E}_j(t) dt \|_S \lesssim 1, \ j = 1, 2 \ .$$

The statement of Proposition 3.3.1 says that if the remainder $\mathcal{E}^t$ has inputs bounded in Z and slightly growing in S then $\mathcal{E}^t$ reproduces the same growth in S and even decays in Z. To prove this proposition, we first present several reductions by performing a decomposition of the non-linearity as

$$\sum_{A,B,C-\text{dyadic}} \mathcal{N}^t[Q_A F(t), Q_B G(t), Q_C H(t)] \ .$$

3.3.1 The High Frequency Estimates

In this subsection, we are going to prove a decay estimate on the non-linearity $\mathcal{N}^t[Q_A F(t), Q_B G(t), Q_C H(t)]$ for $t \sim T$, $T \geq 1$, in the regime $\max(A, B, C) \geq T^{\frac{1}{6}}$. In the case when two inputs have high frequencies, we can simply conclude by using energy estimates, while in the case when the highest frequency is much higher than the others, we invoke the bilinear refinements of the Strichartz estimate on $\mathbb{R}$.

Lemma 3.3.1. *[6] Assume that $\lambda/10 \geq \mu \geq 1$ and that $u(t) = \mathrm{e}^{it\partial_{xx}} u_0$, $v(t) = \mathrm{e}^{it\partial_{xx}} v_0$. Then, we have the bound*

$$\|Q_\lambda u \overline{Q_\mu v}\|_{L^2_{x,t}(\mathbb{R} \times \mathbb{R})} \lesssim \lambda^{-\frac{1}{2}} \|u_0\|_{L^2_x(\mathbb{R})} \|v_0\|_{L^2_x(\mathbb{R})}. \tag{3.3.5}$$

One may refer to [6] for the proof.

Slight modifications of the proof of the corresponding result in [26, Lemma 3.2] lead to the following estimates. We reproduce the proof here for the readers' convenience.

Lemma 3.3.2. *Assume that $T \geq 1$. The following estimates hold uniformly in T :*

$$\Big\| \sum_{\substack{A,B,C \\ \max(A,B,C) \geq T^{\frac{1}{6}}}} \mathcal{N}^t[Q_A F, Q_B G, Q_C H] \Big\|_{Y^s} \lesssim T^{-\frac{5}{4}} \|F\|_S \|G\|_S \|H\|_S, \;\; s > 1, \;\; \forall t \geq T/4, \tag{3.3.6}$$

$$\Big\| \sum_{\substack{A,B,C \\ \max(A,B,C) \geq T^{\frac{1}{6}}}} \int_{\frac{T}{2}}^{T} \mathcal{N}^t[Q_A F(t), Q_B G(t), Q_C H(t)] dt \Big\|_S \lesssim T^{-\frac{1}{50}} \|F\|_{X_T} \|G\|_{X_T} \|H\|_{X_T}, \tag{3.3.7}$$

$$\Big\| \sum_{\substack{A,B,C \\ \max(A,B,C) \geq T^{\frac{1}{6}}}} \int_{\frac{T}{2}}^{T} \mathcal{N}^t[Q_A F(t), Q_B G(t), Q_C H(t)] dt \Big\|_{S^+} \lesssim T^{-\frac{1}{50}} \|F\|_{X_T^+} \|G\|_{X_T^+} \|H\|_{X_T^+}. \tag{3.3.8}$$

Démonstration. Let us begin with the first inequality. Let $K \in L^2_{x,y}$, then we need to bound

$$\begin{aligned} I_K &= \big\langle K, \; \mathcal{N}^t[Q_A F, Q_B G, Q_C H] \big\rangle \\ &\leq \left| \int_{\mathbb{R}\times\mathbb{T}} e^{it\mathcal{A}}(Q_A F) \cdot \overline{e^{it\mathcal{A}}(Q_B G)} \cdot e^{it\mathcal{A}}(Q_C H) \cdot \overline{e^{it\mathcal{A}}(K)} \right| \;. \end{aligned}$$

By Sobolev embedding, we see that

$$\begin{aligned} & \left| \int_{\mathbb{R}\times\mathbb{T}} \mathrm{e}^{it\mathcal{A}}(Q_A F) \cdot \overline{\mathrm{e}^{it\mathcal{A}}(Q_B G)} \cdot \mathrm{e}^{it\mathcal{A}}(Q_C H) \cdot \overline{\mathrm{e}^{it\mathcal{A}}(K)} \right| \\ \lesssim \;& \|\mathrm{e}^{it\mathcal{A}} Q_A F\|_{L^6_{x,y}} \|\mathrm{e}^{it\mathcal{A}} Q_B G\|_{L^6_{x,y}} \|\mathrm{e}^{it\mathcal{A}} Q_C H\|_{L^6_{x,y}} \|K\|_{L^2_{x,y}} \\ \lesssim \;& \|\mathrm{e}^{it\mathcal{A}} Q_A F\|_{H^s_{x,y}} \|\mathrm{e}^{it\mathcal{A}} Q_B G\|_{H^s_{x,y}} \|\mathrm{e}^{it\mathcal{A}} Q_C H\|_{H^s_{x,y}} \|K\|_{L^2_{x,y}} \\ = \;& \|Q_A F\|_{H^s_{x,y}} \|Q_B G\|_{H^s_{x,y}} \|Q_C H\|_{H^s_{x,y}} \|K\|_{L^2_{x,y}} \\ \lesssim \;& (ABC)^{-13+s} \|Q_A F\|_{H^{13}_{x,y}} \|Q_B G\|_{H^{13}_{x,y}} \|Q_C H\|_{H^{13}_{x,y}} \|K\|_{L^2_{x,y}} \;, \end{aligned}$$

with $s > 2/3$. Then by duality, taking $s = 1$, we have

$$\begin{aligned} & \Big\| \mathcal{N}^t[Q_A F, Q_B G, Q_C H] \Big\|_{L^2_{x,y}} \\ & \quad \lesssim (ABC)^{-12} \|Q_A F\|_{H^{13}_{x,y}} \|Q_B G\|_{H^{13}_{x,y}} \|Q_C H\|_{H^{13}_{x,y}} \\ & \quad \lesssim (ABC)^{-12} \|Q_A F\|_S \|Q_B G\|_S \|Q_C H\|_S \;. \end{aligned} \tag{3.3.9}$$

Then By(3.2.16),

$$
\begin{aligned}
&\Big\| \sum_{\substack{A,B,C \\ \max(A,B,C)\geq T^{\frac{1}{6}}}} \mathcal{N}^t[Q_AF, Q_BG, Q_CH] \Big\|_{Y^s} \\
&\lesssim \sum_{\substack{A,B,C \\ \max(A,B,C)\geq T^{\frac{1}{6}}}} \Big\|\mathcal{N}^t[Q_AF, Q_BG, Q_CH]\Big\|_{Y^s} \\
&\lesssim \sum_{\substack{A,B,C \\ \max(A,B,C)\geq T^{\frac{1}{6}}}} \Big\|\mathcal{N}^t[Q_AF, Q_BG, Q_CH]\Big\|_{S}^{3/4} \Big\|\mathcal{N}^t[Q_AF, Q_BG, Q_CH]\Big\|_{L^2}^{1/4} \\
&\lesssim T^{-3/4} \sum_{\substack{A,B,C \\ \max(A,B,C)\geq T^{\frac{1}{6}}}} (ABC)^{-3}\|Q_AF\|_S\|Q_BG\|_S\|Q_CH\|_S \\
&\lesssim T^{-5/4}\|F\|_S\|G\|_S\|H\|_S \ ,
\end{aligned}
$$

where in the third inequality we used Lemma 3.2.2 and (3.3.9).

For the other two estimates, we must be more careful. First of all, we will split the set $\{(A,B,C):\ \max(A,B,C)\geq T^{\frac{1}{6}}\}$ into two parts Λ and its relative complement Λ^c. Here the set Λ is defined as $\Lambda := \Big\{(A,B,C):\ \mathrm{med}(A,B,C)\leq T^{\frac{1}{6}}/16,\ \max(A,B,C)\geq T^{\frac{1}{6}}\Big\}$, with $\mathrm{med}(A,B,C)$ denote the second largest dyadic number among (A,B,C).

Let us start with the case $(A,B,C)\in\Lambda^c$, we claim

$$
\Big\| \sum_{(A,B,C)\in\Lambda^c} \mathcal{N}^t[Q_AF, Q_BG, Q_CH]\Big\|_{S^{(+)}} \lesssim T^{-\frac{11}{6}}\|F\|_{S^{(+)}}\|G\|_{S^{(+)}}\|H\|_{S^{(+)}} \tag{3.3.10}
$$

By Lemma 3.6.2, we only need to control $\Big\|\sum_{(A,B,C)\in\Lambda^c}\mathcal{N}^t[Q_AF, Q_BG, Q_CH]\Big\|_{L^2}$, the main strategy is similar to the proof above, but this time we should not lose derivatives on all of the F, G, H, let us check the condition (3.6.4). Let $K\in L^2_{x,y}$, then we need to bound

$$
\begin{aligned}
I_K &= \Big\langle K, \sum_{(A,B,C)\in\Lambda^c} \mathcal{N}^t[Q_AF, Q_BG, Q_CH]\Big\rangle \\
&\leq \sum_{(A,B,C)\in\Lambda^c} \Big| \int_{\mathbb{R}\times\mathbb{T}} \mathrm{e}^{it\mathcal{A}}(Q_AF)\cdot\overline{\mathrm{e}^{it\mathcal{A}}(Q_BG)}\cdot \mathrm{e}^{it\mathcal{A}}(Q_CH)\cdot\overline{\mathrm{e}^{it\mathcal{A}}(K)}\Big| \\
&\lesssim \sum_{(A,B,C)\in\Lambda^c} \|Q_AF\|_{L^2_{x,y}}\|\mathrm{e}^{it\mathcal{A}}Q_BG\|_{L^\infty_{x,y}}\|\mathrm{e}^{it\mathcal{A}}Q_CH\|_{L^\infty_{x,y}}\|K\|_{L^2_{x,y}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{(A,B,C)\in\Lambda^c} \|Q_A F\|_{L^2_{x,y}} \|Q_B G\|_{H^2_{x,y}} \|Q_C H\|_{H^2_{x,y}} \|K\|_{L^2_{x,y}} \\
&\lesssim \sum_{(A,B,C)\in\Lambda^c} (BC)^{-11} \|Q_A F\|_{L^2_{x,y}} \|Q_B G\|_{H^{13}_{x,y}} \|Q_C H\|_{H^{13}_{x,y}} \|K\|_{L^2_{x,y}} \\
&\lesssim \Big(\sum_{(A,B,C)\in\Lambda^c} (\mathrm{med}(A,B,C))^{-11} \Big) \|F\|_{L^2_{x,y}} \|G\|_{H^{13}_{x,y}} \|H\|_{H^{13}_{x,y}} \|K\|_{L^2_{x,y}} \\
&\lesssim T^{-11/6} \|F\|_{L^2_{x,y}} \|K\|_{L^2_{x,y}} \|G\|_S \|H\|_S \ ,
\end{aligned}
$$

then by duality,

$$
\Big\| \sum_{(A,B,C)\in\Lambda^c} \mathcal{N}^t[Q_A F, Q_B G, Q_C H] \Big\|_{L^2} \lesssim T^{-11/6} \|F\|_{L^2_{x,y}} \|G\|_S \|H\|_S \ . \tag{3.3.11}
$$

The inequality above holds by replacing F with G, H, then we get (3.3.10) by applying Lemma 3.6.2.

Now we turn to the case $(A,B,C) \in \Lambda$, we are to show

$$
\begin{aligned}
&\| \sum_{\substack{A,B,C \\ (A,B,C)\in\Lambda}} \int_{\frac{T}{2}}^{T} \mathcal{N}^t[Q_A F(t), Q_B G(t), Q_C H(t)] dt \|_{S^{(+)}} \\
&\qquad \lesssim T^{-\frac{1}{50}} \|F\|_{X_T^{(+)}} \|G\|_{X_T^{(+)}} \|H\|_{X_T^{(+)}} \ .
\end{aligned}
\tag{3.3.12}
$$

We will only prove the case with norms S and X_T, the proof of the case with S^+, X_T^+ is similar. The main tool of this part is the bilinear Strichartz estimate from Lemma 3.3.1. We consider a decomposition

$$
[T/4, 2T] = \bigcup_{j\in J} I_j \ , \ I_j = [jT^{\frac{9}{10}}, (j+1)T^{\frac{9}{10}}] = [t_j, t_{j+1}] \ , \ \#J \lesssim T^{\frac{1}{10}} \tag{3.3.13}
$$

and consider $\chi \in C_c^\infty(\mathbb{R})$, $\chi \geq 0$ such that $\chi(s) = 0$ if $|s| \geq 2$ and

$$
\sum_{k\in\mathbb{Z}} \chi(s-k) \equiv 1 \ .
$$

The left hand-side of (3.3.12) can be estimated by $C(E_1 + E_2)$, where

$$
\begin{aligned}
E_1 = \Big\| \sum_{j\in J} \sum_{(A,B,C)\in\Lambda} \int_{\frac{T}{2}}^{T} & \chi\big(\frac{t}{T^{\frac{9}{10}}} - j\big) \\
& \Big(\mathcal{N}^t[Q_A F(t), Q_B G(t), Q_C H(t)] - \mathcal{N}^t[Q_A F(t_j), Q_B G(t_j), Q_C H(t_j)] \Big) dt \Big\|_S
\end{aligned}
$$

and

$$
E_2 = \Big\| \sum_{j\in J} \sum_{(A,B,C)\in\Lambda} \int_{\frac{T}{2}}^{T} \chi\big(\frac{t}{T^{\frac{9}{10}}} - j\big) \mathcal{N}^t[Q_A F(t_j), Q_B G(t_j), Q_C H(t_j)] dt \Big\|_S \ .
$$

Notice that $F(t_j),\ G(t_j),\ H(t_j)$ do not depend on t.

Let us start to estimate E_1,

$$E_1 \leq \sum_{j\in J} \int_{\frac{T}{2}}^{T} \chi\big(\frac{t}{T^{\frac{9}{10}}} - j\big) E_{1,j}(t)dt \tag{3.3.14}$$

with

$$E_{1,j}(t) := \Big\| \sum_{(A,B,C)\in\Lambda} \Big(\mathcal{N}^t[Q_A F(t), Q_B G(t), Q_C H(t)] - \mathcal{N}^t[Q_A F(t_j), Q_B G(t_j), Q_C H(t_j)]\Big) \Big\|_S .$$

Denote by $Q_+ := Q_{\geq T^{\frac{1}{6}}}$ and $Q_- := Q_{\leq T^{\frac{1}{6}}/16}$, then due to the structure of Λ, one of A, B, C is larger than $T^{\frac{1}{6}}$ and the other two are smaller than $T^{\frac{1}{6}}/16$, we decompose

$$\sum_{(A,B,C)\in\Lambda} \mathcal{N}^t[Q_A F, Q_B G, Q_C H] = \mathcal{N}^t[Q_+F, Q_-G, Q_-H] + \mathcal{N}^t[Q_-F, Q_+G, Q_-H] + \mathcal{N}^t[Q_-F, Q_-G, Q_+H] .$$

We rearrange the terms in $E_{1,j}$ two by two, and rewrite each pair as follows

$$\begin{aligned}
&\mathcal{N}^t[Q_+F(t), Q_-G(t), Q_-H(t)] - \mathcal{N}^t[Q_+F(t_j), Q_-G(t_j), Q_-H(t_j)]\\
&\quad = \mathcal{N}^t[Q_+(F(t) - F(t_j)), Q_-G, Q_-H(t)] + \mathcal{N}^t[Q_+F(t_j), Q_-(G(t) - Gt_j)), Q_-H(t)]\\
&\quad + \mathcal{N}^t[Q_+F(t_j), Q_-G(t_j), Q_-(H(t) - H(t_j))] ,
\end{aligned}$$

then by Lemma 3.2.2, and the boundedness of $Q_\pm$ on $S^{(+)}$, we see that

$$\|\mathcal{N}^t[Q_+(F(t) - F(t_j)), Q_-G, Q_-H(t)]\|_S \lesssim (1+|t|)^{-1}\|F(t) - F(t_j)\|_S\|G(t)\|_S\|H(t)\|_S ,$$

We bound the other terms similarly, and finally we have an estimate on $E_{1,j}$,

$$\begin{aligned}
E_{1,j}(t) \leq (1+|t|)^{-1}\Big[&\|F(t) - F(t_j)\|_S\|G(t)\|_S\|H(t)\|_S\\
&+ \|F(t_j)\|_S\|G(t) - G(t_j)\|_S\|H(t)\|_S\\
&+ \|F(t_j)\|_S\|G(t_j)\|_S\|H(t) - H(t_j)\|_S\Big] .
\end{aligned} \tag{3.3.15}$$

Since $|t - t_j| \leq T^{\frac{9}{10}}$,

$$\|F(t) - F(t_j)\|_S \leq \int_{t_j}^{t} \|\partial_t F(\theta)\|_S d\theta \leq T^{\frac{9}{10}} \sup_t \|\partial_t F(t)\|_S .$$

Notice that this is the advantage of introducing the partition of time interval provided by χ. Comparing with the definition of X_T (see (3.2.11)), we have

$$\begin{aligned}
\|F(t) - F(t_j)\|_S &\leq T^{-\frac{1}{10}+3\delta}\|F\|_{X_T} ,\\
\|F(t)\|_S &\leq T^{\delta}\|F\|_{X_T} .
\end{aligned}$$

Therefore,

$$E_{1,j} \lesssim T^{-\frac{11}{10}+5\delta} \|F\|_{X_T} \|G\|_{X_T} \|H\|_{X_T} \ ,$$

then

$$E_1 \lesssim \int_{T/2}^{T} \sum_{j\in J} \chi(\frac{t}{T^{\frac{9}{10}}} - j) E_{1,j(t)} dt \lesssim T^{-\frac{1}{10}+5\delta} \|F\|_{X_T} \|G\|_{X_T} \|H\|_{X_T} \ .$$

We now turn to E_2, recall

$$E_2 = \Big\| \sum_{j\in J} \sum_{(A,B,C)\in\Lambda} \int_{\frac{T}{2}}^{T} \chi(\frac{t}{T^{\frac{9}{10}}} - j) \mathcal{N}^t [Q_A F(t_j), Q_B G(t_j), Q_C H(t_j)] dt \Big\|_S \ ,$$

with $Q_A F(t_j), Q_B G(t_j), Q_C H(t_j)$ do not depend on t. Denoting

$$E_{2,j}^{A,B,C} = \Big\| \int_{\frac{T}{2}}^{T} \chi(\frac{t}{T^{\frac{9}{10}}} - j) \mathcal{N}^t [Q_A F(t_j), Q_B G(t_j), Q_C H(t_j)] dt \ \Big\|_S \ ,$$

then

$$E_2 \leq \sum_{j\in J} \sum_{(A,B,C)\in\Lambda} E_{2,j}^{A,B,C} \ .$$

We claim

$$\begin{aligned} &\Big\| \int_{\frac{T}{2}}^{T} \chi(\frac{t}{T^{\frac{9}{10}}} - j) \mathcal{N}^t [Q_A F^a, Q_B F^b, Q_C F^c] dt \Big\|_{L^2_{x,y}} \\ &\lesssim (\max(A,B,C))^{-1} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{L^2_{x,y}} \|F^\beta\|_S \|F^\gamma\|_S . \end{aligned} \tag{3.3.16}$$

Then by Lemma 3.6.2, $\|E_{2,j}^{A,B,C}\|_S \lesssim (\max(A,B,C))^{-1} \|F\|_S \|G\|_S \|H\|_S$, the estimate for E_2 will come out by summing up. Let us prove (3.3.16), assuming $K \in L^2_{x,y}$, we consider with functions F^a, F^b, F^c independent on t,

$$\begin{aligned} I_K &= \sum_{p-q+r-s=0} \mathrm{e}^{it\omega} \langle K_p, \int_{\frac{T}{2}}^{T} \chi(\frac{t}{T^{\frac{9}{10}}} - j) \mathcal{N}^t [Q_A F^a_q, Q_B F^b_r, Q_C F^c_s] dt \rangle_{L^2_x \times L^2_x} \\ &= \sum_{p-q+r-s=0} \mathrm{e}^{it\omega} \int_{\frac{T}{2}}^{T} \int_{\mathbb{R}\times\mathbb{T}} \chi(\frac{t}{T^{\frac{9}{10}}} - j) \mathrm{e}^{it\partial_{xx}} (Q_A F^a_q) \overline{\mathrm{e}^{it\partial_{xx}} (Q_B F^b_r)} \mathrm{e}^{it\partial_{xx}} (Q_C F^c_s) \overline{\mathrm{e}^{it\partial_{xx}} K_p} dx dt \end{aligned}$$

where we may assume that $K = Q_D K$, $D \simeq \max(A,B,C)$. Without loss of generality, we assume $A = \max(A,B,C)$, then by Hölder's inequality,

$$\begin{aligned} &\Big| \int_{\frac{T}{2}}^{T} \int_{\mathbb{R}\times\mathbb{T}} \chi(\frac{t}{T^{\frac{9}{10}}} - j) \mathrm{e}^{it\partial_{xx}} (Q_A F^a_q) \overline{\mathrm{e}^{it\partial_{xx}} (Q_B F^b_r)} \mathrm{e}^{it\partial_{xx}} (Q_C F^c_s) \overline{\mathrm{e}^{it\partial_{xx}} Q_D K_p} dx dt \Big| \\ &\leq \|\mathrm{e}^{it\partial_{xx}} (Q_A F^a_q) \overline{\mathrm{e}^{it\partial_{xx}} (Q_B F^b_r)}\|_{L^2_{x,t}} \|\mathrm{e}^{it\partial_{xx}} (Q_C F^c_s) \overline{\mathrm{e}^{it\partial_{xx}} Q_D K_p}\|_{L^2_{x,t}} \ , \end{aligned}$$

since $A \geq 16B$, $D \geq 16C$, applying the bilinear Strichartz estimate from Lemma 3.3.1 below, we then have

$$\begin{aligned} \|\mathrm{e}^{it\partial_{xx}} (Q_A F^a_q) \overline{\mathrm{e}^{it\partial_{xx}} (Q_B F^b_r)}\|_{L^2_{x,t}} &\lesssim A^{-1/2} \|F^a_q\|_{L^2_x} \|F^b_s\|_{L^2_x} \\ \|\mathrm{e}^{it\partial_{xx}} (Q_C F^c_s) \overline{\mathrm{e}^{it\partial_{xx}} Q_D K_p}\|_{L^2_{x,t}} &\lesssim D^{-1/2} \|F^c_r\|_{L^2_x} \|K_p\|_{L^2_x} \ . \end{aligned}$$

Applying Cauchy-Schwarz and (3.2.17) on the summation on the right hand side of I_K, we have

$$\begin{aligned}
I_K &\lesssim \sum_{p-q+r-s=0} (\max(A,B,C))^{-1} \|F_q^a\|_{L_x^2} \|F_r^b\|_{L_x^2} \|F_s^c\|_{L_x^2} \|K_p\|_{L_x^2} \\
&\lesssim (\max(A,B,C))^{-1} \Big\| \sum_{p=q-r+s} \|F_q^a\|_{L_x^2} \|F_r^b\|_{L_x^2} \|F_s^c\|_{L_x^2} \Big\|_{\ell_p^2} \|K_p\|_{L_{x,y}^2} \\
&\lesssim (\max(A,B,C))^{-1} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{L_{x,y}^2} \sum_p \|F_p^\beta\|_{L_x^2} \sum_p \|F_p^\gamma\|_{L_x^2} \|K\|_{L_{x,y}^2} \\
&\lesssim (\max(A,B,C))^{-1} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \Big(\|F^\alpha\|_{L_{x,y}^2} \sum_p \big(|p|^{-s}|p|^s \|\|F_p^\beta\|_{L_x^2}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \sum_p \big(|p|^{-s}|p|^s \|F^\gamma\|_{L_x^2}\big) \|K\|_{L_{x,y}^2} \Big) \\
&\lesssim (\max(A,B,C))^{-1} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{L_{x,y}^2} \|F^\beta\|_S \|F^\gamma\|_S \|K\|_{L_{x,y}^2} ,
\end{aligned}$$

where we took $s > 1/2$. The result (3.3.16) turns out by duality. Applying Lemma 3.6.2, we get

$$E_{2,j}^{A,B,C} \lesssim (\max(A,B,C))^{-1} \|F\|_S \|G\|_S \|H\|_S ,$$

then

$$E_2 \leq \sum_{j\in J} \sum_{(A,B,C)\in\Lambda} E_{2,j}^{A,B,C} \lesssim \#J \sum_{(A,B,C)\in\Lambda} (\max(A,B,C))^{-1} \|F\|_S \|G\|_S \|H\|_S .$$

Without loss of generality, we assume $A = \max(A,B,C)$, then

$$\sum_{(A,B,C)\in\Lambda} (\max(A,B,C))^{-1} = \Big(\sum_{A\geq T^{1/6}} A^{-1} \Big) \big(\#\{B : B \leq T^{1/6}/16\}\big)^2 \lesssim T^{-1/6+\delta}$$

while using the definition (3.2.11),

$$\|F(t_j)\|_S \|G(t_j)\|_S \|H(t_j)\|_S \leq T^{3\delta} \|F\|_{X_T} \|G\|_{X_T} \|H\|_{X_T} ,$$

thus

$$E_2 \lesssim T^{-1/15+\delta} \|F\|_{X_T} \|G\|_{X_T} \|H\|_{X_T} , \tag{3.3.17}$$

which is a stronger version of (3.3.12). The proof of Lemma 3.3.2 is complete. $\square$

Thus we may suppose that the x frequencies of F, G, H are $\lesssim T^{\frac{1}{6}}$. It is natural to introduce the first decomposition

$$\mathcal{N}^t[F,G,H] = \mathcal{N}_0^t[F,G,H] + \widetilde{\mathcal{N}}^t[F,G,H] , \tag{3.3.18}$$

$$\mathcal{F}\mathcal{N}_0^t(\xi,p) := \sum_{(p,q,r,s)\in\Gamma_0} \mathcal{F}_x\big(\mathcal{I}^t[F_q,G_r,H_s]\big)(\xi) . \tag{3.3.19}$$

3.3.2 The fast oscillations

Firstly, we present another elementary estimate here.

Lemma 3.3.3. *Let $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$ with $1 \leq p, q, r, s \leq \infty$, then*

$$\| \int_{\mathbb{R}^3} \mathrm{e}^{ix\xi} m(\eta, \kappa) \widehat{f}(\xi - \eta) \widehat{\overline{g}}(\xi - \eta - \kappa) \widehat{h}(\xi - \kappa) d\eta d\kappa d\xi \big\|_{L^p_x} \lesssim \|\mathcal{F}^{-1} m\|_{L^1(\mathbb{R}^2)} \|f\|_{L^q} \|g\|_{L^r} \|h\|_{L^s}.$$

Démonstration.

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \mathrm{e}^{ix\xi} m(\eta, k) \widehat{f}(\xi - \eta) \widehat{\overline{g}}(\xi - \eta - k) \widehat{h}(\xi - k) d\eta d\kappa d\xi \\ &= \int_{\mathbb{R}^3 \times \mathbb{R}^2} \left(\int_{\mathbb{R}^3} \mathrm{e}^{i\xi(x - \alpha + \beta - \gamma)} \mathrm{e}^{-i\eta(y - \alpha + \beta)} \mathrm{e}^{i\kappa(z + \beta - \gamma)} d\xi d\eta d\kappa \right) \\ &\qquad\qquad\qquad\qquad \mathcal{F}^{-1} m(y, z) f(\alpha) \overline{g}(\beta) h(\gamma) dy dz d\alpha d\beta d\gamma \\ &= \int_{\mathbb{R}^2} \mathcal{F}^{-1} m(y, z) f(x - z) \overline{g}(x - y - z) h(x - y) dy dz \ , \end{aligned}$$

then

$$\begin{aligned} \|I\|_{L^p_x} &\leq \int_{\mathbb{R}^3} |\mathcal{F}^{-1} m(y, z)| \|f \overline{g} h\|_{L^p_x} dy dz \\ &= \|\mathcal{F}^{-1} m\|_{L^1(\mathbb{R}^2)} \|f \overline{g} h\|_{L^p(\mathbb{R})} \\ &\leq \|\mathcal{F}^{-1} m\|_{L^1(\mathbb{R}^2)} \|f\|_{L^q} \|g\|_{L^r} \|h\|_{L^s} \ , \end{aligned}$$

the last inequality comes from the Hölder's inequality and the assumption $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$. □

Remark 3.3.1. *Similar result holds for the case $m = m(\xi, \eta, \kappa)$, one may refer to [26, Lemma 7.5].*

The main purpose of this subsection is to estimate of $\widetilde{\mathcal{N}}^t$.

Lemma 3.3.4. *Let $1 \leq T \leq T^*$. Assume that F, G, $H : \mathbb{R} \to S$ satisfy (3.3.3) and*

$$F = Q_{\leq T^{1/6}} F, \quad G = Q_{\leq T^{1/6}} G, \quad H = Q_{\leq T^{1/6}} H \ .$$

Then we can write

$$\widetilde{\mathcal{N}}^t[F(t), G(t), H(t)] = \widetilde{\mathcal{E}}^t_1[F(t), G(t), H(t)] + \mathcal{E}^t_2[F(t), G(t), H(t)] \ ,$$

and if we set $\widetilde{\mathcal{E}}_1(t) := \widetilde{\mathcal{E}}^t_1[F(t), G(t), H(t)]$ and $\mathcal{E}_2(t) := \mathcal{E}^t_2[F(t), G(t), H(t)]$ then it holds that, uniformly in $1 \leq T \leq T^*$,

$$T^{1+2\delta} \sup_{T/4 \leq t \leq T^*} \|\widetilde{\mathcal{E}}_1(t)\|_S \lesssim 1 \ , \quad T^{1/10} \sup_{T/4 \leq t \leq T^*} \|\mathcal{E}_3(t)\|_S \lesssim 1 \ ,$$

where $\mathcal{E}_2(t) = \partial_t \mathcal{E}_3(t)$. Assuming in addition that (3.3.4) holds we have

$$T^{1+2\delta} \sup_{T/4 \leq t \leq T^*} \|\widetilde{\mathcal{E}}_1(t)\|_{S^+} \lesssim 1 \ , \qquad T^{1/10} \sup_{T/4 \leq t \leq T^*} \|\mathcal{E}_3(t)\|_{S^+} \lesssim 1 \ .$$

Démonstration. To prove this lemma, we start by decomposing $\widetilde{\mathcal{N}}^t$ along the non-resonant level sets as follows : Set

$$F^a = Q_{\leq T^{1/6}} F^a, \quad F^b = Q_{\leq T^{1/6}} F^b, \quad F^c = Q_{\leq T^{1/6}} F^c \ ,$$

$$\mathcal{F}\widetilde{\mathcal{N}}^t[F^a, F^b, F^c](\xi, p) = \sum_{\omega \neq 0} \sum_{(p,q,r,s)\in\Gamma_\omega} \mathrm{e}^{it\omega} \left(\mathcal{O}_1^t[F_q^a, F_r^b, F_s^c](\xi) + \mathcal{O}_2^t[F_q^a, F_r^b, F_s^c](\xi) \right) \ , \tag{3.3.20}$$

$$\mathcal{O}_1^t[f^a, f^b, f^c](\xi) := \int_{\mathbb{R}^2} \mathrm{e}^{2it\eta\kappa}(1 - \varphi(t^{\frac{1}{4}}\eta\kappa))\widehat{f^a}(\xi - \eta)\overline{\widehat{f^b}}(\xi - \eta - \kappa)\widehat{f^c}(\xi - \kappa)d\eta d\kappa \ ,$$
$$\mathcal{O}_2^t[f^a, f^b, f^c](\xi) := \int_{\mathbb{R}^2} \mathrm{e}^{2it\eta\kappa}\varphi(t^{\frac{1}{4}}\eta\kappa)\widehat{f^a}(\xi - \eta)\overline{\widehat{f^b}}(\xi - \eta - \kappa)\widehat{f^c}(\xi - \kappa)d\eta d\kappa \ .$$

We may rewrite for $\omega \neq 0$,

$$\begin{aligned} \mathrm{e}^{it\omega}\mathcal{O}_2^t[f^a, f^b, f^c] &= \partial_t \left(\frac{\mathrm{e}^{it\omega}}{i\omega} \mathcal{O}_2^t[f^a, f^b, f^c] \right) - \frac{\mathrm{e}^{it\omega}}{i\omega} \left(\partial_t \mathcal{O}_2^t \right) [f^a, f^b, f^c] \\ &- \frac{\mathrm{e}^{it\omega}}{i\omega} \mathcal{O}_2^t[\partial_t f^a, f^b, f^c] - \frac{\mathrm{e}^{it\omega}}{i\omega} \mathcal{O}_2^t[f^a, \partial_t f^b, f^c] - \frac{\mathrm{e}^{it\omega}}{i\omega} \mathcal{O}_2^t[f^a, f^b, \partial_t f^c] \\ &:= \partial_t \left(\frac{\mathrm{e}^{it\omega}}{i\omega} \mathcal{O}_2^t[f^a, f^b, f^c] \right) + \mathrm{e}^{it\omega} \mathcal{L}^t[f^a, f^b, f^c] \ , \end{aligned} \tag{3.3.21}$$

where

$$\left(\partial_t \mathcal{O}_2^t \right) [f^a, f^b, f^c] := \int_{\mathbb{R}^2} \partial_t \left(\mathrm{e}^{2it\eta\kappa} \varphi(t^{\frac{1}{4}}\eta\kappa) \right) \widehat{f^a}(\xi - \eta)\overline{\widehat{f^b}}(\xi - \eta - \kappa)\widehat{f^c}(\xi - \kappa)d\eta d\kappa \ .$$

Thus we define $\mathcal{E}_2^t[F^a, F^b, F^c] = \partial_t \mathcal{E}_3^t[F^a, F^b, F^c]$ with

$$\mathcal{F}\mathcal{E}_3^t[F^a, F^b, F^c](\xi, p) := \sum_{\omega \neq 0} \sum_{(p,q,r,s)\in\Gamma_\omega} \left(\frac{\mathrm{e}^{it\omega}}{i\omega} \mathcal{O}_2^t[F_q^a, F_r^b, F_s^c] \right) \ , \tag{3.3.22}$$

and define $\widetilde{\mathcal{E}}_1^t$ with $\mathcal{O}_1^t$ and the last four terms in (3.3.21),

$$\mathcal{F}\widetilde{\mathcal{E}}_1^t(\xi, p) := \sum_{\omega \neq 0} \sum_{(p,q,r,s)\in\Gamma_\omega} \mathrm{e}^{it\omega} \left(\mathcal{O}_1^t[F_q^a, F_r^b, F_s^c] + \mathcal{L}^t[F_q^a, F_r^b, F_s^c] \right) \ . \tag{3.3.23}$$

1. Estimation of $\mathcal{E}_3(t)$. We define the multiplier appearing in the definition of $\mathcal{O}_2^t$ by

$$m(\eta, \kappa) := \varphi(t^{\frac{1}{4}}\eta\kappa)\varphi((10T)^{\frac{-1}{6}}\eta)\varphi((10T)^{\frac{-1}{6}}\kappa) \ .$$

>From Lemma 3.3.5 at the end of this subsection, it is bounded by $\|\mathcal{F}_{\eta\kappa}\widetilde{m}\|_{L^1(\mathbb{R}^2)} \lesssim t^{\frac{\delta}{100}}$. Applying Lemma 3.3.3, we get

$$\|\mathcal{O}_2^t[f^a, f^b, f^c]\|_{L^2_\xi} \lesssim (1 + |t|)^{\frac{\delta}{100}} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|f^\alpha\|_{L^2_x} \|\mathrm{e}^{it\partial_{xx}} f^\beta\|_{L^\infty_x} \|\mathrm{e}^{it\partial_{xx}} f^\gamma\|_{L^\infty_x} \ .$$

Then

$$
\begin{aligned}
\|\mathcal{E}_3(t)\|_{L^2_{x,y}} &\lesssim \Big\| \sum_{\omega\neq 0} \sum_{(p,q,r,s)\in\Gamma_\omega} \left(\frac{\mathrm{e}^{it\omega}}{i\omega} \mathcal{O}_2^t[F_q^a, F_r^b, F_s^c] \right) \Big\|_{L^2_x \ell^2_p} \\
&\lesssim (1+|t|)^{\frac{\delta}{100}} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \Big\| \sum_{p-q+r-s=0} \|F_q^\alpha\|_{L^2_x} \|\mathrm{e}^{it\partial_{xx}} F_r^\beta\|_{L^\infty_x} \|\mathrm{e}^{it\partial_{xx}} F_s^\gamma\|_{L^\infty_x} \Big\|_{\ell^2_p}
\end{aligned}
$$

using (3.2.17)

$$
\lesssim (1+|t|)^{\frac{\delta}{100}} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{L^2_{x,y}} \sum_r \|\mathrm{e}^{it\partial_{xx}} F_r^\beta\|_{L^\infty_x} \sum_s \|\mathrm{e}^{it\partial_{xx}} F_s^\gamma\|_{L^\infty_x}
$$

using (3.2.22)

$$
\begin{aligned}
&\lesssim (1+|t|)^{-1+\frac{\delta}{100}} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \Big(\|F^\alpha\|_{L^2_{x,y}} \sum_r \left(\|F_r^\beta\|_{L^2_x}^{1/2} \|xF_r^\beta\|_{L^2_x}^{1/2} \right) \\
&\qquad \cdot \sum_s \left(\|F_s^\gamma\|_{L^2_x}^{1/2} \|xF_s^\gamma\|_{L^2_x}^{1/2} \right) \Big) .
\end{aligned}
$$

Noticing that for the last inequality, we have

$$
\begin{aligned}
\sum_r (|a_r|^{1/2} |b_r|^{1/2}) &\leq \sum_r (|a_r|^{1/2} |r|^{\theta} |r|^{-\theta} |b_r|^{1/2}) \\
&\leq \|a_r\|_{h_r^{2\theta}}^{1/4} (\sum_r |r|^{-2\theta})^{1/2} \|b_r\|_{\ell^2_r} \\
&\lesssim \|a_r\|_{h_r^{2\theta}}^{1/2} \|b_r\|_{\ell^2_r}
\end{aligned}
$$

with $\theta > 1/2$. Then

$$
\begin{aligned}
\|\mathcal{E}_3(t)\|_{L^2_{x,y}} &\lesssim (1+|t|)^{-1+\frac{\delta}{100}} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{L^2_{x,y}} \|F^\beta\|_{H^{2\theta}_{x,y}}^{1/2} \|xF^\beta\|_{L^2_{x,y}}^{1/2} \|F^\gamma\|_{H^{2\theta}_{x,y}}^{1/2} \|xF^\gamma\|_{L^2_{x,y}}^{1/2} \\
&\lesssim (1+|t|)^{-1+\frac{\delta}{100}} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{L^2_{x,y}} \|F^\beta\|_S \|F^\gamma\|_S \ . \qquad (3.3.24)
\end{aligned}
$$

Therefore, an application of Lemma 3.6.2 shows that the S norms of S_3 is controlled as follows,

$$
\|\mathcal{E}_3(t)\|_S \lesssim (|T|)^{-1+\frac{\delta}{100}} \|F^a\|_S \|F^b\|_S \|F^c\|_S \lesssim (|T|)^{-1+\frac{\delta}{100}} \ , \qquad (3.3.25)
$$

the last inequality comes from (3.3.3). Combining with inequality (3.3.4), we can also gain

$$
\|S_3(t)\|_{S^+} \lesssim (|T|)^{-1+\frac{\delta}{100}} \ . \qquad (3.3.26)
$$

2. Estimation of $\widetilde{\mathcal{E}}_1(t)$. Again, we need to control the L^2 norm first, and then the S norm. $\widetilde{\mathcal{E}}_1(t)$ is composed by two parts, one is from $\mathcal{O}_1^t$, and the other one $\mathcal{L}^t$ is from the last four terms in (3.3.21),

$$
\mathcal{F}\widetilde{\mathcal{E}}_1^t[F^a, F^b, F^c](\xi, p) := \sum_{\omega\neq 0} \sum_{(p,q,r,s)\in\Gamma_\omega} \mathrm{e}^{it\omega} \left(\mathcal{O}_1^t[F_q^a, F_r^b, F_s^c] + \mathcal{L}^t[F_q^a, F_r^b, F_s^c] \right) \ , \qquad (3.3.27)
$$

with

$$i\omega\mathcal{L}^t[f^a,f^b,f^c] := -\left(\partial_t\mathcal{O}_2^t\right)[f^a,f^b,f^c]-\mathcal{O}_2^t[\partial_t f^a,f^b,f^c]-\mathcal{O}_2^t[f^a,\partial_t f^b,f^c]-\mathcal{O}_2^t[f^a,f^b,\partial_t f^c]\,.$$

The term $\sum\limits_{\omega\neq 0}\sum_{(p,q,r,s)\in\Gamma_\omega} \mathrm{e}^{it\omega}\mathcal{L}^t[F_q^a,F_r^b,F_s^c]$ can be estimated similarly as $\|\mathcal{E}_3(t)\|_S$. Actually, we may gain a better estimate here, since for the first term, we can get an extra $T^{-1/4}$ which comes from the t derivative of the multiplier, while for the other three terms, by the definition of X_T norm, we have $\|\partial_t F\|_S \le T^{-1+3\delta}\|F\|_{X_T}$. Let us focus on

$$\sum_{\omega\neq 0}\sum_{(p,q,r,s)\in\Gamma_\omega} \mathrm{e}^{it\omega}\mathcal{O}_1^t[F_q^a,F_r^b,F_s^c]\,.$$

We claim that

$$\Big\|\sum_{\omega\neq 0}\sum_{(p,q,r,s)\in\Gamma_\omega} \mathrm{e}^{it\omega}\mathcal{O}_1^t[F_q^a,F_r^b,F_s^c]\Big\|_{L^2} \lesssim T^{-1-\delta}\min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}}\|F^\alpha\|_{L^2_{x,y}}\|F^\beta\|_S\|F^\gamma\|_S\,. \tag{3.3.28}$$

As we did for $\mathcal{O}_2^t$, we still have

$$\|\mathcal{O}_1^t[f^a,f^b,f^c]\|_{L^2} \lesssim (1+|t|)^{\delta/100}\min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}}\|f^\alpha\|_{L^2}\|\mathrm{e}^{it\partial_{xx}}f^\beta\|_{L^\infty}\|\mathrm{e}^{it\partial_{xx}}f^\gamma\|_{L^\infty}\,. \tag{3.3.29}$$

We then need to estimate $\|\mathrm{e}^{it\partial_{xx}}f\|_{L^\infty_x}$. We notice that for all $\frac{1}{2}<\alpha\le 1$,

$$\|\mathrm{e}^{it\partial_{xx}}f\|_{L^\infty(\mathbb{R})} \lesssim \langle t\rangle^{-\frac12}\|f\|_{L^1(\mathbb{R})} \lesssim \langle t\rangle^{-\frac12}\|\langle x\rangle^{-\alpha}\langle x\rangle^{\alpha}f\|_{L^1(\mathbb{R})} \lesssim \langle t\rangle^{-\frac12}\|\langle x\rangle^{\alpha}f\|_{L^2(\mathbb{R})}\,, \tag{3.3.30}$$

we may take $\alpha=7/9$, then for f supported on $|x|\ge R$,

$$\|\mathrm{e}^{it\partial_{xx}}f\|_{L^\infty} \lesssim \langle t\rangle^{-\frac12}R^{-1/9}\|\langle x\rangle^{8/9}f\|_{L^2}\,. \tag{3.3.31}$$

Therefore, we decompose $f=f_c+f_e$ with $f_c(x):=\varphi(\frac{x}{T^{1/4}})f(x)$, then

$$\mathcal{O}_1^t[f^a,f^b,f^c]=\mathcal{O}_1^t[f_c^a+f_e^a,f_c^b+f_e^b,f_c^c+f_e^c]\,.$$

then by (3.3.31), if one of f^a,f^b,f^c is supported on $|x|\ge 2T^{1/4}$, for example, $f^b=f_e^b$, then

$$\begin{aligned}\|\mathcal{O}_1^t[f^a,f_e^b,f^c]\|_{L^2_x} = &\lesssim (1+|t|)^{\delta/100}\|f^a\|_{L^2}\|\mathrm{e}^{it\partial_{xx}}f_e^b\|_{L^\infty}\|\mathrm{e}^{it\partial_{xx}}f^c\|_{L^\infty}\\ &\lesssim (1+|t|)^{\delta/100}\|f^a\|_{L^2}\|\mathrm{e}^{it\partial_{xx}}f_e^b\|_{L^\infty}\|\mathrm{e}^{it\partial_{xx}}f^c\|_{L^\infty}\\ &\lesssim T^{-1-1/36+\delta/100}\|f^a\|_{L^2}\|\langle x\rangle^{8/9}f_e^b\|_{L^2}\|\langle x\rangle^{7/9}f^c\|_{L^2}\,,\end{aligned}$$

in the last inequality comes from (3.3.30) and (3.3.31). Then using (3.2.17),

$$\begin{aligned}&\Big\|\sum_{\omega\neq 0}\sum_{(p,q,r,s)\in\Gamma_\omega} \mathrm{e}^{it\omega}\mathcal{O}_1^t[F_q^a,F_{r,e}^b,F_s^c]\Big\|_{L^2}\\ &\lesssim T^{-1-1/36+\delta/100}\|F^a\|_{L^2}\sum_r\|\langle x\rangle^{8/9}F_r^b\|_{L^2_x}\sum_s\|\langle x\rangle^{7/9}F_s^c\|_{L^2_x}\,.\end{aligned} \tag{3.3.32}$$

For $0 < \alpha < 1$,

$$\sum_r \|x^\alpha F_r\|_{L^2_x} \lesssim \|F\|_S \ , \tag{3.3.33}$$

indeed,

$$\begin{aligned}\sum_r \|\langle x\rangle^\alpha F_r\|_{L^2_x} &= \sum_r \|(\langle x\rangle F_r)^\alpha F_r^{1-\alpha}\|_{L^2_x} \le \sum_r \|\langle x\rangle F_r\|_{L^2}^\alpha \|F_r\|_{L^2}^{1-\alpha} \\ &\le \sum_r \|\langle x\rangle F_r\|_{L^2}^\alpha \langle r\rangle^s \|F_r\|_{L^2}^{1-\alpha}\langle r\rangle^{-s} \le \|\langle x\rangle F\|_{L^2_{x,y}}^\alpha \|F\|^{1-\alpha}_{H^{\frac{s}{1-\alpha}}_{x,y}} \le \|F\|_S \ ,\end{aligned}$$

with $s > 1/2$. Thus

$$\Big\|\sum_{\omega\neq 0}\sum_{(p,q,r,s)\in\Gamma_\omega} \mathrm{e}^{it\omega}\mathcal{O}_1^t[F_q^a, F_{r,e}^b, F_s^c]\Big\|_{L^2} \lesssim T^{-1-1/36+\delta/100}\|F^a\|_{L^2}\|F^b\|_S\|F^c\|_S \ . \tag{3.3.34}$$

Let us turn to the case $\mathcal{O}_1^t[f^a, f^b_c, f^c_c]$. By replacing $\mathrm{e}^{2it\eta\kappa}$ by $(2it\eta)^{-1}\partial_\kappa(\mathrm{e}^{2it\eta\kappa})$, we can rewrite $\mathcal{O}_1^t$ as

$$\begin{aligned}\mathcal{O}_1^t[f^a, f^b, f^c](\xi) &= \int_{\mathbb{R}^2} \mathrm{e}^{2it\eta\kappa}(1-\varphi(t^{\frac14}\eta\kappa))\widehat{f^a}(\xi-\eta)\overline{\widehat{f^b}}(\xi-\eta-\kappa)\widehat{f^c}(\xi-\kappa)d\eta d\kappa \\ &= \int_{\mathbb{R}^2} (2it\eta)^{-1}\partial_\kappa(\mathrm{e}^{2it\eta\kappa})(1-\varphi(t^{\frac14}\eta\kappa))\widehat{f^a}(\xi-\eta)\overline{\widehat{f^b}}(\xi-\eta-\kappa)\widehat{f^c}(\xi-\kappa)d\eta d\kappa \\ &= \int_{\mathbb{R}^2} (2it\eta)^{-1}\mathrm{e}^{2it\eta\kappa}\partial_\kappa\big((1-\varphi(t^{\frac14}\eta\kappa))\widehat{f^a}(\xi-\eta)\overline{\widehat{f^b}}(\xi-\eta-\kappa)\widehat{f^c}(\xi-\kappa)\big)d\eta d\kappa \ .\end{aligned} \tag{3.3.35}$$

Firstly, it is easy to deal with the case when the κ derivative falls on $1-\varphi$, which turns out to be

$$(2i)^{-1}t^{-3/4}\int_{\mathbb{R}^2} \mathrm{e}^{2it\eta\kappa}\varphi'(t^{\frac14}\eta\kappa)\widehat{f^a}(\xi-\eta)\overline{\widehat{f^b}}(\xi-\eta-\kappa)\widehat{f^c}(\xi-\kappa)d\eta d\kappa \ ,$$

then we get the required estimate with the similar strategy we used to estimate $\mathcal{O}_2^t$ since φ' admits similar properties as φ.

For the other case, we calculate the case when κ derivative falls on f^b for example, which is denoted by $\mathcal{O}_{1,b}$,

$$\begin{aligned}\mathcal{O}_{1,b} &:= \int_{\mathbb{R}^2} (2it\eta)^{-1}\mathrm{e}^{2it\eta\kappa}(1-\varphi(t^{\frac14}\eta\kappa))\widehat{f^a}(\xi-\eta)\partial_\kappa\big(\overline{\widehat{f^b}}(\xi-\eta-\kappa)\big)\widehat{f^c}(\xi-\kappa)d\eta d\kappa \\ &= \int_{\mathbb{R}^2} (2it\eta)^{-1}\mathrm{e}^{2it\eta\kappa}(1-\varphi(t^{\frac14}\eta\kappa))\widehat{f^a}(\xi-\eta)\overline{\widehat{xf^b}}(\xi-\eta-\kappa)\widehat{f^c}(\xi-\kappa)d\eta d\kappa \ .\end{aligned} \tag{3.3.36}$$

Noticing that on the support of the integration, $|t||\eta| \gtrsim |t|^{-3/4}|\kappa|^{-1} \gtrsim T^{-7/12}$, we still have an L^2 estimate

$$\|\mathcal{O}_{1,b}\|_{L^2_\xi} \lesssim T^{-7/12+\frac{\delta}{100}}\|f^a\|_{L^2}\cdot\|\mathrm{e}^{it\partial_{xx}}(xf^b)\|_{L^\infty}\cdot\|\mathrm{e}^{it\partial_{xx}}f^c\|_{L^\infty} \ . \tag{3.3.37}$$

By (3.3.30), for f supported on $|x| \leq T^{1/4}$, we have

$$\|e^{it\partial_{xx}} xf\|_{L^\infty} \lesssim \langle t\rangle^{-\frac{1}{2}} T^{1/4} \|\langle x\rangle^{7/9} f\|_{L^2} \ . \tag{3.3.38}$$

using (3.3.30) and (3.3.38),

$$\begin{aligned}
\|\mathcal{O}_{1,b}\|_{L^2_\xi} &\lesssim T^{-7/12+\frac{\delta}{100}} \|f^a\|_{L^2} \cdot \|e^{it\partial_{xx}}(xf^b_c)\|_{L^\infty} \cdot \|e^{it\partial_{xx}} f^c\|_{L^\infty} \\
&\lesssim T^{-4/3+\frac{\delta}{100}} \|f^a\|_{L^2_x} \|\langle x\rangle^{7/9} f^b\|_{L^2_x} \|\langle x\rangle^{7/9} f^c\|_{L^2_x} \ .
\end{aligned}$$

Once again we use (3.3.33),

$$\begin{aligned}
&\Big\| \sum_{\omega\neq 0} \sum_{(p,q,r,s)\in\Gamma_\omega} e^{it\omega} \mathcal{O}^t_1[F^a_q, F^b_r, F^c_s] \Big\|_{L^2} \\
&\lesssim T^{-4/3+\frac{\delta}{100}} \Big\| \sum_{\omega\neq 0} \sum_{(p,q,r,s)\in\Gamma_\omega} \|F^a_q\|_{L^2_x} \|\langle x\rangle^{7/9} F^b_r\|_{L^2_x} \|\langle x\rangle^{7/9} F^c_s\|_{L^2_x} \Big\|_{\ell^2_p} \\
&\lesssim T^{-4/3+\frac{\delta}{100}} \|F^a\|_{L^2_{x,y}} \|F^b\|_S \|F^c\|_S \ .
\end{aligned} \tag{3.3.39}$$

By replacing F^a by F^b or F^c, we proved (3.3.28) and then the estimate of $\widetilde{\mathcal{E}}_1(t)$. □

Lemma 3.3.5. *[26, Remark 3.5] For $T > 1$, $\varphi \in C^\infty_c(\mathbb{R})$, $\varphi(x) = 1$ when $|x| \leq 1$ and $\varphi(x) = 0$ when $|x| \geq 2$, we define for $T/2 \leq t \leq T$,*

$$\widetilde{m}(\eta,\kappa) := \varphi(t^{\frac{1}{4}}\eta\kappa)\varphi((10T)^{\frac{-1}{6}}\eta)\varphi((10T)^{\frac{-1}{6}}\kappa) \ .$$

Then $\|\mathcal{F}_{\eta\kappa}\widetilde{m}\|_{L^1(\mathbb{R}^2)} \lesssim t^{\frac{\delta}{100}}$.

Démonstration.

$$\|\mathcal{F}_{\eta\kappa}\widetilde{m}\|_{L^1(\mathbb{R}^2)} = \|I(x_1,x_2)\|_{L^1_{x_1,x_2}} \ ,$$

where

$$I(x_1,x_2) = \int_{\mathbb{R}^2} e^{ix_1\eta} e^{ix_2\kappa} \varphi(S\eta\kappa)\varphi(\eta)\varphi(\kappa)\, d\eta d\kappa, \quad S \approx T^{\frac{7}{12}} \ .$$

Then one may show that

$$|I(x_1,x_2)| + |x_1 I(x_1,x_2)| + |x_2 I(x_1,x_2)| \lesssim 1, \quad |x_1x_2 I(x_1,x_2)| \lesssim \log(1+T) \ .$$

Indeed,

$$\begin{aligned}
|x_1 I(x_1,x_2)| &= \left| \int_{\mathbb{R}^2} \frac{1}{i}\partial_\eta(e^{ix_1\eta}) e^{ix_2\kappa} \varphi(S\eta\kappa)\varphi(\eta)\varphi(\kappa)\, d\eta d\kappa \right| \\
&= \left| \int_{\mathbb{R}^2} e^{ix_1\eta} e^{ix_2\kappa} [S\kappa\varphi'(S\eta\kappa)\varphi(\eta)\varphi(\kappa) + \varphi(S\eta\kappa)\varphi'(\eta)\varphi(\kappa)]\, d\eta d\kappa \right| \\
&\lesssim 1 + \left| \int_{\mathbb{R}^2} e^{ix_1\eta} e^{ix_2\kappa} (S\kappa\varphi'(S\eta\kappa)\varphi(\eta)\varphi(\kappa))\, d\eta d\kappa \right| \ .
\end{aligned}$$

Notice that $|S\eta\kappa| \le 2$, then the second term turns out to be

$$\left|\int_{\mathbb{R}^2} \mathrm{e}^{ix_1\eta}\mathrm{e}^{ix_2\kappa}(S\kappa\varphi'(S\eta\kappa)\varphi(\eta)\varphi(\kappa))d\eta d\kappa\right| \lesssim \int_{D:=\{|S\eta\kappa|,|\eta|,|\kappa|\le 2\}} |S\kappa| d\eta d\kappa \lesssim 1.$$

Thus we get the first inequality, and we use the similar strategy to prove the second one.

$$\begin{aligned}
|x_1x_2I(x_1,x_2)| &\lesssim \int_{\mathbb{R}^2} |\partial_\eta\partial_\kappa\big(\varphi(S\eta\kappa)\varphi(\eta)\varphi(\kappa)\big)|d\eta d\kappa \\
&\lesssim \int_D |S\kappa\varphi'(S\eta\kappa)\varphi(\eta)\varphi'(\kappa)| + |S\kappa S\eta\varphi''(S\eta\kappa)\varphi(\eta)\varphi(\kappa)| \\
&\quad + |S\eta\varphi'(S\eta\kappa)\varphi'(\eta)\varphi(\kappa)| + |\varphi(S\eta\kappa)\varphi'(\eta)\varphi'(\kappa)| + |S\varphi'(S\eta\kappa)\varphi(\eta)\varphi(\kappa)|d\eta d\kappa \\
&\lesssim (\int_0^{T^{-7/12}}\int_0^2 + \int_{T^{-7/12}}^2\int_0^{\frac{2T^{-7/12}}{\kappa}})[1+|S\kappa|+|S\eta|+|S\kappa S\eta|+S]d\eta d\kappa \\
&\lesssim \log(1+T)\ .
\end{aligned}$$

Then

$$(1+|x_1|)(1+|x_2|)|I(x_1,x_2)| \lesssim \log(1+T)\ .$$

One also have a polynomial in T bound

$$(1+|x_1|^2)(1+|x_2|^2)|I(x_1,x_2)| \lesssim T^{7/12}\ .$$

Therefore by interpolation one obtains that for every $0<\varepsilon<7/12$, there exists $\kappa>1/2$ such that

$$|I(x_1,x_2)| \lesssim (1+T)^\varepsilon(1+|x_1|^2)^{-\kappa}(1+|x_2|^2)^{-\kappa}\ .$$

We hence deduce that $\|\mathcal{F}_{\eta\kappa}\widetilde{m}\|_{L^1(\mathbb{R}^2)} \lesssim t^{\frac{\delta}{100}}$. □

3.3.3 The Resonant Level sets

We now turn to the contribution of the resonant part in (3.3.18),

$$\mathcal{F}\mathcal{N}_0^t[F,G,H](\xi,p) = \sum_{(p,q,r,s)\in\Gamma_0} \mathcal{F}_x\mathcal{I}^t[F_q(t),G_r(t),H_s(t)](\xi).$$

This term yields the main contribution in Proposition 3.3.1 and in particular is responsible for the slowest $1/t$ decay. We show that it gives rise to a contribution which grows slowly in S, S^+ and that it can be well approximated by the resonant system in the Z norm.

In this subsection, we will bound quantities in terms of

$$\|F\|_{\widetilde{Z}_t} := \|F\|_Z + (1+|t|)^{-\delta}\|F\|_S\ ,$$

so that $F(t)$ remains uniformly bounded in $\widetilde{Z}_t$ under the assumption of Proposition 3.3.1 due to the definition of X_T and $\widetilde{Z}_t$ norm. Our main statement of this subsection is as follows.

Lemma 3.3.6. *Let $t \geq 1$. There holds that*

$$\|\mathcal{N}_0^t[F^a, F^b, F^c]\|_S \lesssim (1+|t|)^{-1} \sum_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{\widetilde{Z}_t} \cdot \|F^\beta\|_{\widetilde{Z}_t} \cdot \|F^\gamma\|_S \tag{3.3.40}$$

and

$$\begin{aligned}\|\mathcal{N}_0^t[F^a, F^b, F^c]\|_{S^+} \lesssim &(1+|t|)^{-1} \sum_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{\widetilde{Z}_t} \cdot \|F^\beta\|_{\widetilde{Z}_t} \cdot \|F^\gamma\|_{S^+} \\ &+ (1+|t|)^{-1+2\delta} \sum_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{\widetilde{Z}_t} \cdot \|F^\beta\|_S \cdot \|F^\gamma\|_S.\end{aligned} \tag{3.3.41}$$

Moreover,

$$\|\mathcal{N}_0^t[F, G, H] - \frac{\pi}{t}\mathcal{R}[F, G, H]\|_{Y^s} \lesssim (1+|t|)^{-1-20\delta} \|F\|_S \|G\|_S \|H\|_S \ . \tag{3.3.42}$$

and

$$\|\mathcal{N}_0^t[F, G, H] - \frac{\pi}{t}\mathcal{R}[F, G, H]\|_S \lesssim (1+|t|)^{-1-20\delta} \|F\|_{S^+} \|G\|_{S^+} \|H\|_{S^+} \ . \tag{3.3.43}$$

In addition, we also have

$$\|\mathcal{R}[F^a, F^b, F^c]\|_S \lesssim \sum_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{\widetilde{Z}_t} \cdot \|F^\beta\|_{\widetilde{Z}_t} \cdot \|F^\gamma\|_S \tag{3.3.44}$$

$$\begin{aligned}\|\mathcal{R}[F^a, F^b, F^c]\|_{S^+} \lesssim &\sum_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{\widetilde{Z}_t} \cdot \|F^\beta\|_{\widetilde{Z}_t} \cdot \|F^\gamma\|_{S^+} \\ &+ (1+|t|)^{2\delta} \sum_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{\widetilde{Z}_t} \cdot \|F^\beta\|_S \cdot \|F^\gamma\|_S.\end{aligned} \tag{3.3.45}$$

Démonstration. As before, we will study the L^2 norm and then apply Lemma 3.6.2 to get the $S^{(+)}$ norm estimate. Using (3.2.17), we have

$$\begin{aligned}\|\mathcal{N}_0^t[F^a, F^b, F^c]\|_{L^2_{x,y}} &\leq \Big\| \sum_{(p,q,r,s)\in\Gamma_0} |\mathrm{e}^{it\partial_{xx}} F^a_q| \cdot |\mathrm{e}^{it\partial_{xx}} F^b_r| \cdot |\mathrm{e}^{it\partial_{xx}} F^c_s| \Big\|_{\ell^2_p L^2_x} \\ &\lesssim \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{L^2_{x,y}} \sum_p \|\mathrm{e}^{it\partial_{xx}} F^\beta_p\|_{L^\infty_x} \sum_p \|\mathrm{e}^{it\partial_{xx}} F^\gamma_p\|_{L^\infty_x} \ .\end{aligned}$$

To calculate $\sum_p \|\mathrm{e}^{it\partial_{xx}} F_p\|_{L^\infty_x}$, we start with the following estimate for $|t| > 1$,

$$\Big|\mathrm{e}^{it\partial_{xx}} f(x) - c\frac{\mathrm{e}^{-ix^2/(4t)}}{\sqrt{t}} \widehat{f}(-\frac{x}{2t})\Big| \lesssim |t|^{-3/4} \|xf\|_{L^2} \ , c \text{ is a constant.} \tag{3.3.46}$$

One may refer to [26, Lemma 7.3] for the proof of (3.3.46). Then

$$|\mathrm{e}^{it\partial_{xx}} f(x)| \lesssim |t|^{-1/2} \sup_\xi |\widehat{f}(\xi)| + |t|^{-3/4} \|xf\|_{L^2} \ . \tag{3.3.47}$$

Then

$$\sum_{|p|\le t^{1/8}} \|e^{it\partial_{xx}} F_p\|_{L^\infty_x} \lesssim t^{-1/2} \sup_\xi \sum_p |\widehat{F_p}(\xi)| + t^{-3/4} \sum_{|p|\le t^{1/8}} \|xF_p\|_{L^2}$$
$$\lesssim t^{-1/2}\|F\|_Z + t^{-5/8}\|xF\|_S \ ,$$

while

$$\sum_{|p|>t^{1/8}} \|e^{it\partial_{xx}} F_p\|_{L^\infty_x} \lesssim \sum_{|p|>t^{1/8}} \|F_p\|_{H^1}$$
$$= \sum_{|p|>t^{1/8}} (1+|p|^2)^{N/2} \|F_p\|_{H^1} (1+|p|^2)^{-N/2}$$
$$\lesssim t^{-\frac{2N-1}{16}} \|F\|_{H^{N+1}_{x,y}} \ ,$$

therefore

$$\|\mathcal{N}_0^t[F^a, F^b, F^c]\|_{L^2_{x,y}} \lesssim (1+|t|)^{-1} \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{L^2_{x,y}} \|F^\beta\|_{\widetilde{Z}_t} \|F^\gamma\|_{\widetilde{Z}_t} \ . \tag{3.3.48}$$

Apply the first part of Lemma 3.6.2, we get (3.3.40). The proof of (3.3.41) follows from the second part of Lemma 3.6.2, and we only need to check $\widetilde{Z}_t$ norm satisfies (3.6.6). Due to the definition of S^+, we only need to prove the following inequality,

$$\|(1-\partial_{xx})^4 F\|_Z + \|xF\|_Z \lesssim T^{-\delta}\|F\|_{S^+} + T^{2\delta}\|F\|_S \ . \tag{3.3.49}$$

Indeed, following the proof of (3.2.14), we are able to show

$$\|(1-\partial_{xx})^4 F\|_Z \lesssim \|F\|_S \ ,$$

thus we only need to prove (3.3.49) for $\|xF\|_Z$. Since $H^s(\mathbb{T}) \subset B^1$ with $s>1$, then

$$\|xF\|_Z^2 = \sup_\xi (1+|\xi|^2)^2 \|\widehat{xF}\|^2_{B^1_y}$$
$$\lesssim \sup_M (1+M^2) \sum_p (1+|p|^2)^s |\mathcal{F}_x Q_M(xF_p)|^2 \ .$$

We notice that for any $M, |p| \neq 0$, denote $R = (1+M^2)(1+|p|^2)^s T^{2\delta}$, and we decompose $xF_p(x)$ into two parts

$$x(1-\varphi(\frac{x}{R}))F_p \text{ and } x\varphi(\frac{x}{R})F_p \ ,$$

then

$$\|xF\|_Z^2 \lesssim \sup_M (1+M^2) \sum_p (1+|p|^2)^s \|\mathcal{F}_x Q_M\{x(1-\varphi(\frac{x}{R}))F_p\}\|^2_{L^\infty_\xi} := \mathrm{I}$$
$$+ \sup_M (1+M^2) \sum_p (1+|p|^2)^s \|\mathcal{F}_x Q_M\{x\varphi(\frac{x}{R})F_p\}\|_{L^\infty_\xi} := \mathrm{II} \ .$$

$$
\begin{aligned}
\mathrm{I} &\lesssim \sup_M (1+M^2) \sum_p (1+|p|^2)^s \|xF_p\|^2_{L^1_x(|x|>R)} \\
&\lesssim \sup_M (1+M^2) \sum_p (1+|p|^2)^s R^{-1} \|x^2 F_p\|^2_{L^2} \\
&\lesssim T^{-2\delta} \|x^2 F\|_{L^2} \leq T^{-2\delta} \|F\|_{S^+} \ ,
\end{aligned}
$$

while

$$
\begin{aligned}
\mathrm{II} &\lesssim \sup_M (1+M^2) \sum_p (1+|p|^2)^s \|Q_M\{x\varphi(\frac{x}{R})F_p\}\|^2_{L^1_x(|x|\leq R)} \\
&\lesssim \sup_M (1+M^2) \sum_p (1+|p|^2)^s R^2 \|F_p\|_{L^2_x} \|xF_p\|_{L^2_x} \\
&\lesssim \sum_p T^{4\delta} \|F_p\|_{L^2_x} \|xF_p\|_{L^2_x} \lesssim T^{4\delta} \|F\|^2_S \ ,
\end{aligned}
$$

thus we proved (3.3.49), the estimate on $\mathcal{R}$ is the same.

Now we turn to the proof of the error estimates (3.3.42) and (3.3.43). We first decompose the functions as we did for estimating $\mathcal{O}_1^t$,

$$
F = F_c + F_e, \quad \text{with } F_c \text{ compactly supported as } F_c = \varphi(\frac{x}{t^{1/4}})F \ ,
$$

and reduce the problem to the estimates on F_c, G_c, H_c. We start with the L^2 estimates of

$$
\mathcal{N}_0^t[F,G,H] - \mathcal{N}_0^t[F_c,G_c,H_c] \quad \text{and} \quad \mathcal{R}[F,G,H] - \mathcal{R}[F_c,G_c,H_c] \ ,
$$

without loss of generalities, it suffices to consider $\mathcal{N}_0^t[F_e,G,H]$ and $\frac{1}{t}\mathcal{R}[F_e,G,H]$. Indeed, using (3.3.48) and the definition of F_e,

$$
\begin{aligned}
\|\mathcal{N}_0^t[F_e,G,H]\|_{L^2} + \frac{1}{t}\|\mathcal{R}[F_e,G,H]\|_{L^2} &\lesssim (1+|t|)^{-1} \|F_e\|_{L^2} \|G\|_S \|H\|_S \\
&\lesssim (1+|t|)^{-5/4} \|F\|_S \|G\|_S \|H\|_S \ ,
\end{aligned} \tag{3.3.50}
$$

while

$$
\|\mathcal{N}_0^t[F_e,G,H]\|_S + \frac{1}{t}\|\mathcal{R}[F_e,G,H]\|_S \lesssim (1+|t|)^{-1} \|F\|_S \|G\|_S \|H\|_S \ . \tag{3.3.51}
$$

Thus by (3.2.16), we are able to bound

$$
\begin{aligned}
&\|\mathcal{N}_0^t[F,G,H] - \mathcal{N}_0^t[F_c,G_c,H_c]\|_{Y^s} + \frac{1}{t}\|\mathcal{R}[F,G,H] - \mathcal{R}[F_c,G_c,H_c]\|_{Y^s} \\
&\quad \lesssim (1+|t|)^{-17/16} \|F\|_S \|G\|_S \|H\|_S \ .
\end{aligned} \tag{3.3.52}
$$

For the S norm estimate, we use (3.3.51) again,

$$
\begin{aligned}
\|\mathcal{N}_0^t[F_e,G,H]\|_S + \frac{1}{t}\|\mathcal{R}[F_e,G,H]\|_S &\lesssim (1+|t|)^{-1} \|F_e\|_S \|G\|_S \|H\|_S \\
&\lesssim (1+|t|)^{-5/4} \|F\|_{S^+} \|G\|_{S^+} \|H\|_{S^+} \ .
\end{aligned} \tag{3.3.53}
$$

Therefore, we only need to show the inequalities below to complete our proof of this lemma,

$$\|\mathcal{N}_0^t[F_c,G_c,H_c]-\frac{\pi}{t}\mathcal{R}[F_c,G_c,H_c]\|_{Y^s}\lesssim(1+|t|)^{-1-20\delta}\|F\|_S\|G\|_S\|H\|_S\ , \tag{3.3.54}$$

$$\|\mathcal{N}_0^t[F_c,G_c,H_c]-\frac{\pi}{t}\mathcal{R}[F_c,G_c,H_c]\|_S\lesssim(1+|t|)^{-1-20\delta}\|F\|_{S^+}\|G\|_{S^+}\|H\|_{S^+}\ . \tag{3.3.55}$$

For abbreviation, we assume for the rest part of proof, $F=F_c, G=G_c, H=H_c$.

$$\begin{aligned}&\mathcal{F}\big(\mathcal{N}_0^t[F,G,H]-\frac{\pi}{t}\mathcal{R}[F,G,H]\big)(\xi,p)\\ &=\sum_{(p,q,r,s)\in\Gamma_0}\int_{\mathbb{R}^2}e^{it2\eta\kappa}\widehat{F_q}(\xi-\eta)\overline{\widehat{G_r}}(\xi-\eta-\kappa)\widehat{H_s}(\xi-\kappa)d\kappa d\eta-\frac{\pi}{t}\widehat{F_q}(\xi)\overline{\widehat{G_r}}(\xi)\widehat{H_s}(\xi)\ .\end{aligned} \tag{3.3.56}$$

Rewrite the integration part,

$$\begin{aligned}&\int_{\mathbb{R}^2}\mathrm{e}^{it2\eta\kappa}\widehat{F_q}(\xi-\eta)\overline{\widehat{G_r}}(\xi-\eta-\kappa)\widehat{H_s}(\xi-\kappa)d\kappa d\eta\\ &=\int_{\mathbb{R}^3}F_q(x_1)\overline{G_r}(x_2)H_s(x_3)\int_{\mathbb{R}^2}\mathrm{e}^{it2\eta\kappa}\mathrm{e}^{-ix_1(\xi-\eta)-ix_2(\xi-\eta-\kappa)-ix_3(\xi-\kappa)}d\kappa d\eta dx_1dx_2dx_3\\ &=\frac{1}{2t}\int_{\mathbb{R}^3}F_q(x_1)\overline{G_r}(x_2)H_s(x_3)\mathrm{e}^{-i\xi(x_1-x_2+x_3)}\mathrm{e}^{-i\frac{x_1-x_2}{\sqrt{2t}}\frac{x_3-x_2}{\sqrt{2t}}}\\ &\qquad\qquad\left\{\int_{\mathbb{R}^2}\mathrm{e}^{i\left[\eta+\frac{x_3-x_2}{\sqrt{2t}}\right]\left[\kappa+\frac{x_1-x_2}{\sqrt{2t}}\right]}d\eta d\kappa\right\}dx_1dx_2dx_3\\ &=\frac{\pi}{t}\int_{\mathbb{R}^3}F_q(x_1)\overline{G_r}(x_2)H_s(x_3)\mathrm{e}^{-i\xi(x_1-x_2+x_3)}\mathrm{e}^{-i\frac{x_1-x_2}{\sqrt{2t}}\frac{x_3-x_2}{\sqrt{2t}}}dx_1dx_2dx_3\ ,\end{aligned}$$

then

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\mathrm{e}^{it2\eta\kappa}\widehat{F_q}(\xi-\eta)\overline{\widehat{G_r}}(\xi-\eta-\kappa)\widehat{H_s}(\xi-\kappa)d\kappa d\eta-\frac{\pi}{t}\widehat{F_q}(\xi)\overline{\widehat{G_r}}(\xi)\widehat{H_s}(\xi)\right|\\ &=\frac{\pi}{|t|}\left|\int_{\mathbb{R}^3}F_q(x_1)\overline{G_r}(x_2)H_s(x_3)\mathrm{e}^{-i\xi(x_1-x_2+x_3)}\left(\mathrm{e}^{-i\frac{x_1-x_2}{\sqrt{2t}}\frac{x_3-x_2}{\sqrt{2t}}}-1\right)dx_1dx_2dx_3\right|\\ &\lesssim|t|^{-11/10}\|F_q\|_{L_x^2}\|G_r\|_{L_x^2}\|H_s\|_{L_x^2}\ .\end{aligned}$$

Actually, using the proof above, we may obtain for any integer m,

$$\begin{aligned}&|\xi|^m\left|\int_{\mathbb{R}^2}\mathrm{e}^{it2\eta\kappa}\widehat{F_q}(\xi-\eta)\overline{\widehat{G_r}}(\xi-\eta-\kappa)\widehat{H_s}(\xi-\kappa)d\kappa d\eta-\frac{\pi}{t}\widehat{F_q}(\xi)\overline{\widehat{G_r}}(\xi)\widehat{H_s}(\xi)\right|\\ &\lesssim|t|^{-11/10}\|F_q\|_{H_x^m}\|G_r\|_{L_x^2}\|H_s\|_{L_x^2}\ .\end{aligned} \tag{3.3.57}$$

Due to the definition of Y^s norm (3.2.8) and S norm (3.2.10), and the fact that $H^s(\mathbb{T})$, $s>1$ is an algebra, the proof of (3.3.54) follows from (3.3.57). For (3.3.55), recall that the functions are spectrally compacted supported, then the terms

$$\|\mathcal{N}_0^t[F_c,G_c,H_c]-\frac{\pi}{t}\mathcal{R}[F_c,G_c,H_c]\|_{L_x^2H_y^N}\text{ and }\|x\Big(\mathcal{N}_0^t[F_c,G_c,H_c]-\frac{\pi}{t}\mathcal{R}[F_c,G_c,H_c]\Big)\|_{L_{x,y}^2}$$

are easy to deal with by (3.3.57) and (3.2.17). We should be more careful with the terms admitting x derivatives, since this x derivative may fall on $\varphi(\frac{x}{t^{1/4}})$. Anyhow, since φ' holds the similar properties as φ, (3.3.57) still works, and we are able to get the estimate (3.3.55). The proof is complete. $\square$

3.3.4 Proof of Proposition 3.3.1

Now, we can give the proof of Proposition 3.3.1.

Proof of Proposition 3.3.1. We may firstly decompose the non-linearity $\mathcal{N}^t$ as the high frequency part and then the lower frequency part combined with the resonant and non resonant parts,

$$\begin{aligned}\mathcal{N}^t[F,G,H] = &\sum_{\substack{A,B,C\\ \max(A,B,C)\geq T^{\frac{1}{6}}}} \mathcal{N}^t[Q_AF(t),Q_BG(t),Q_CH(t)]\\ &+\widetilde{\mathcal{N}}^t[Q_{\leq T^{\frac{1}{6}}}F(t),Q_{\leq T^{\frac{1}{6}}}G(t),Q_{\leq T^{\frac{1}{6}}}H(t)]+\mathcal{N}_0^t[Q_{\leq T^{\frac{1}{6}}}F(t),Q_{\leq T^{\frac{1}{6}}}G(t),Q_{\leq T^{\frac{1}{6}}}H(t)]\ .\end{aligned}$$

Then, we rewrite the last term as

$$\begin{aligned}\mathcal{N}_0^t[Q_{\leq T^{\frac{1}{6}}}F(t),Q_{\leq T^{\frac{1}{6}}}G(t),Q_{\leq T^{\frac{1}{6}}}H(t)] &= \frac{\pi}{t}\mathcal{R}[F(t),G(t),H(t)]\\ &+\Big(\mathcal{N}_0^t[Q_{\leq T^{\frac{1}{6}}}F(t),Q_{\leq T^{\frac{1}{6}}}G(t),Q_{\leq T^{\frac{1}{6}}}H(t)]-\frac{\pi}{t}\mathcal{R}[Q_{\leq T^{\frac{1}{6}}}F(t),Q_{\leq T^{\frac{1}{6}}}G(t),Q_{\leq T^{\frac{1}{6}}}H(t)]\Big)\\ &-\frac{\pi}{t}\sum_{\substack{A,B,C\\ \max(A,B,C)\geq T^{\frac{1}{6}}}}\mathcal{R}[Q_AF(t),Q_BG(t),Q_CH(t)]\ .\end{aligned}$$

Finally, we have the formula for the remainder

$$\begin{aligned}\mathcal{E}^t[F,G,H] = &\\ &\sum_{\substack{A,B,C\\ \max(A,B,C)\geq T^{\frac{1}{6}}}} \mathcal{N}^t[Q_AF(t),Q_BG(t),Q_CH(t)]+\widetilde{\mathcal{N}}^t[Q_{\leq T^{\frac{1}{6}}}F(t),Q_{\leq T^{\frac{1}{6}}}G(t),Q_{\leq T^{\frac{1}{6}}}H(t)]\\ &+\Big(\mathcal{N}_0^t[Q_{\leq T^{\frac{1}{6}}}F(t),Q_{\leq T^{\frac{1}{6}}}G(t),Q_{\leq T^{\frac{1}{6}}}H(t)]-\frac{\pi}{t}\mathcal{R}[Q_{\leq T^{\frac{1}{6}}}F(t),Q_{\leq T^{\frac{1}{6}}}G(t),Q_{\leq T^{\frac{1}{6}}}H(t)]\Big)\\ &-\frac{\pi}{t}\sum_{\substack{A,B,C\\ \max(A,B,C)\geq T^{\frac{1}{6}}}}\mathcal{R}[Q_AF(t),Q_BG(t),Q_CH(t)]\ .\end{aligned}$$

Let us exam the terms on the right hand side one by one. The first term contributes to $\mathcal{E}_1$ by Lemma 3.3.2. The second term contains $\mathcal{E}_2$ as it can be written by lemma 3.3.4 as $\widetilde{\mathcal{E}}_1+\mathcal{E}_2$ with $\widetilde{\mathcal{E}}_1$ contributing to $\mathcal{E}_1$. The last two terms contributes to $\mathcal{E}_1$ by Lemma 3.3.6 and its remark. This finishes the proof of Proposition 3.3.1. $\square$

3.4 The Resonant System

In this section, we will study the following resonant system

$$i\partial_t G = \mathcal{R}[G, G, G] \ . \tag{3.4.1}$$

Before further discussions, let us recall a useful result on the structure of the resonances at first.

Lemma 3.4.1. *[12, Lemma 1] Given* $(p_1, p_2, p_3, p_4) \in \Gamma_0$*, namely,*

$$p_1 - p_2 + p_3 - p_4 = 0 \text{ and } |p_1| - |p_2| + |p_3| - |p_4| = 0$$

if and only if at least one of the following properties holds :

1. $\forall j, \ p_j \geq 0$;
2. $\forall j, \ p_j \leq 0$;
3. $p_1 = p_2 \ , \ p_3 = p_4$;
4. $p_1 = p_4 \ , \ p_3 = p_2$.

The following proposition shows us that we are able to get rid of the resonances corresponding to cases (3) and (4), and deduce our resonant system to a decoupling system, which only contains cubic Szegő equations.

Proposition 3.4.1. *Given* $G_0 \in L^2_x H^s_y$*,* $s > 1$*,* $\|G_0\|_{L^2_x H^s_y} = \varepsilon$*,* $\varepsilon > 0$ *and* $G_0(x, y) = -G_0(x, y + \pi)$*. Set* $G^1(t) = \mathrm{e}^{2it\|G_0\|^2_{L^2}} G(t)$ *with* G *as the corresponding solution to the resonant system* (3.4.1)*, then* $G^1(t)$ *satisfies the following cubic Szegő equation,*

$$i\partial_t G^1_\pm = \mathcal{R}_\pm[G^1_\pm, G^1_\pm, G^1_\pm] \ , \tag{3.4.2}$$

where

$$\mathcal{F}_x \mathcal{R}_\pm[G^1_\pm, G^1_\pm, G^1_\pm](\xi, y) = \Pi_\pm(|\widehat{G^1}_\pm|^2 \widehat{G^1}_\pm)(\xi, y) \ , \tag{3.4.3}$$

with $G^1_+ = \Pi_+(G^1) := \sum_{p>0} G^1_p(x) \mathrm{e}^{ipy}$ *and* $G^1_- = \Pi_-(G^1) := \sum_{p<0} G^1_p(x) \mathrm{e}^{ipy}$*.*

Démonstration. The proof of the proposition above is easy. First, by the transformation,

$$G^1(t) = \mathrm{e}^{2it\|G_0\|^2_{L^2}} G(t) \ ,$$

and using the fact that the L^2 norm is conserved, we get our first reduction to the resonant system corresponding to cases (1) and (2). And thanks to our initial condition $G_0(x, y + \pi) = -G_0(x, y)$, we have

$$\mathcal{F}_y G_0(x, p) = 0, \quad p \text{ even numbers} \ ,$$

which insures the decoupling. □

3.4.1 The cubic Szegő equation

Let us begin with a simpler model, a resonant system for a vector $a = \{a_p\}_{p>0}$,

$$i\partial_t a_p(t) = \sum_{(p,q,r,s)\in\Gamma_{0,+}} a_q(t)\overline{a_r(t)}a_s(t) := R_+[a(t),a(t),a(t)]_p \ , \tag{3.4.4}$$

where $\Gamma_{0,+} := \{(p_1,p_2,p_3,p_4) : \ p_1 - p_2 + p_3 - p_4 = 0, p_j > 0 \ \forall j\}$. If we denote $v(t,y) := \sum_{p>0} a_p(t)e^{ipy}$, then v satisfies the following cubic Szegő equation

$$i\partial_t v = \Pi_+(|v|^2 v) \ . \tag{3.4.5}$$

Let us recall more for the cubic Szegő equation (3.4.5), especially the Lax pair structure and its conserved quantities. Gérard and Grellier have showed that the cubic Szegő equation is a completely integrable system with two Lax pairs. One may refer to [11, 13] for more details. To define the Lax pairs, one may need to introduce the Hankel operator H_v and the Toeplitz operator T_b with $v \in H_+^{\frac{1}{2}}(\mathbb{T})$, $b \in L^\infty(\mathbb{T})$,

$$H_v h := \Pi_+(v\bar{h}) \ , T_b h := \Pi_+(bh) \ , h \in L^\infty \ . \tag{3.4.6}$$

We remark that H_v is $\mathbb{C}-$antilinear, and is a Hilbert-Schmidt operator. Now we are able to introduce the Lax pair structure of the cubic Szegő equation (3.4.5),

Theorem 3.4.1. *[11, Theorem 3.1] Let $v \in C(\mathbb{R}, H_+^s(\mathbb{T}))$ for some $s > \frac{1}{2}$. The cubic Szegő equation (3.4.5) has a Lax pair (H_v, B_v), namely, if v solves (3.4.5), then*

$$\frac{dH_v}{dt} = [B_v, H_v] \ , \tag{3.4.7}$$

where $B_v = \frac{i}{2}H_v^2 - iT_{|v|^2}$,

A direct consequence of this Lax pair structure is that the spectrum of the trace class operator H_v^2, is conserved by the evolution, in particular, the trace norm of H_v^2 is conserved by the flow. A theorem by Peller [40, Theorem2, P454], says that the trace norm of a Hankel operator H_v is equivalent to the Besov norm $B^1_{1,1}(\mathbb{T})$ of v. One may also see [12].

3.4.2 Estimation of solutions to the resonant system

We are now able to state a result concerning the long time behavior and stability of the asymptotic system (3.4.1).

Lemma 3.4.2. *For every function G^1, G^2, G^3, the following estimates hold true*

$$\|\mathcal{R}[G^1,G^2,G^3]\|_{L^2_{x,y}} \lesssim \min_{\{j,k,\ell\}=\{1,2,3\}} \|G^j\|_{L^2_{x,y}}\|G^k\|_Z\|G^\ell\|_Z \ , \tag{3.4.8}$$

$$\|\mathcal{R}[G^1,G^2,G^3]\|_Z \lesssim \|G^1\|_Z\|G^2\|_Z\|G^3\|_Z \ , \tag{3.4.9}$$

$$\|\mathcal{R}[G^1,G^2,G^3]\|_S \lesssim \max_{\{j,k,\ell\}=\{1,2,3\}} \|G^j\|_S\|G^k\|_Z\|G^\ell\|_Z \ . \tag{3.4.10}$$

Démonstration. The first inequality comes from (3.2.17). Indeed, by the definition of $\mathcal{R}$,

$$
\begin{aligned}
\|\mathcal{R}[G^1,G^2,G^3]\|_{L^2_{x,y}} &= \Big\| \sum_{\Gamma_{0,+}\cup\Gamma_{0,-}} \widehat{G}^1_q \overline{\widehat{G}^2_r} \widehat{G}^3_s \Big\|_{L^2_\xi \ell^2_p} \\
&\lesssim \min_{\{j,k,\ell\}=\{1,2,3\}} \|G^j\|_{L^2_{x,y}} \|\widehat{G}^k\|_{L^\infty_\xi B^1_y} \|\widehat{G}^\ell\|_{L^\infty_\xi B^1_y} \\
&\lesssim \min_{\{j,k,\ell\}=\{1,2,3\}} \|G^j\|_{L^2_{x,y}} \|G^k\|_Z \|G^\ell\|_Z \ ,
\end{aligned}
$$

Apply Lemma 3.6.2, we get the third inequality. The second inequality comes from the fact that B^1 is an algebra. $\square$

Proposition 3.4.2. *Assume $G_0 \in S^{(+)}$, $\|G_0\|_{S^{(+)}} = \varepsilon$ with ε small enough, and G evolves according to* (3.4.1)*. Then there holds that for $t \geq 1$,*

$$\|G(\pi \ln t)\|_Z \simeq \|G_0\|_Z \ , \tag{3.4.11}$$

$$\|G(\pi \ln t)\|_S \lesssim (1+|t|)^{\delta'} \|G_0\|_S \ , \tag{3.4.12}$$

$$\|G(\pi \ln t)\|_{S^+} \lesssim (1+|t|)^{\delta''} \|G_0\|_{S^+} \ , \tag{3.4.13}$$

with $\delta' \simeq \|G_0\|_Z^2$, $\delta'' \simeq \|G_0\|_S^3 \|G_0\|_Z^{-1}$.

Démonstration. For the first conservation, we use the complete integrability of the cubic Szegő equation, especially its Lax pair and the conservation of the B^1 norm, which is stated in the previous subsection. First, one may use Proposition 3.4.1 to reduce our problem to the cubic Szegő equation, and the transformation we used keeps the Z norm. Then we use Peller's theorem to obtain

$$\|\widehat{G}(\xi,t)\|_{B^1} \simeq \mathrm{Tr}|H_{\widehat{G}(\xi,t)}| \ .$$

Combined with the Lax Pair structure, we have

$$\|\widehat{G}(\xi,t)\|_{B^1} \simeq \mathrm{Tr}|H_{\widehat{G}(\xi,t)}| \simeq \mathrm{Tr}|H_{\widehat{G}_0(\xi)}| \simeq \|\widehat{G}_0(\xi)\|_{B^1} \ .$$

For the second one, taking $\widetilde{G}(t) = G(\pi \ln t)$, then $\widetilde{G}$ satisfies

$$i\partial_t \widetilde{G} = \frac{\pi}{t} \mathcal{R}[\widetilde{G},\widetilde{G},\widetilde{G}] \ . \tag{3.4.14}$$

The main idea is to estimate the $S^{(+)}$ norm of $\mathcal{R}[\widetilde{G},\widetilde{G},\widetilde{G}]$, and then apply the Gronwall's inequality.

Indeed, using (3.4.10),

$$\partial_t \|\widetilde{G}\|_S \lesssim \frac{1}{t} \|\mathcal{R}[\widetilde{G},\widetilde{G},\widetilde{G}]\|_S \lesssim \frac{1}{t} \|\widetilde{G}\|_Z^2 \|\widetilde{G}\|_S \lesssim \frac{1}{t} \|G_0\|_Z^2 \|\widetilde{G}\|_S \ ,$$

thus we get the S norm estimate by Gronwall's inequality.

We now turn to the S^+ norm estimate, by the proof of the estimate (3.3.49), we may gain another more general version,

$$\|(1-\partial_{xx})^4 G\|_Z + \|xG\|_Z \lesssim t^{-\delta'} \|G_0\|_S^{-1} \|G_0\|_Z \|G\|_{S^+} + t^{2\delta''} \|G_0\|_{S_+} \|G_0\|_S \|G_0\|_Z^{-2} \|G\|_S \ ,$$

with $\delta' \lesssim \delta'' \simeq \|G_0\|_S^3 \|G_0\|_Z^{-1}$. We apply the second part of Lemma 3.6.2,

$$\|\mathcal{R}[\widetilde{G}, \widetilde{G}, \widetilde{G}]\|_{S^+} \lesssim \|\widetilde{G}\|_{S^+} \Big(\|G_0\|_Z^2 + t^{-\delta'} \|G_0\|_S^{-1} \|G_0\|_Z \|\widetilde{G}\|_S \Big) \\ + t^{2\delta''} \|G_0\|_{S_+} \|G_0\|_S \|G_0\|_Z^{-1} \|\widetilde{G}\|_S^2 \ ,$$

then plugging the estimate of $\|\widetilde{G}\|_S$,

$$\frac{d}{dt} \|\widetilde{G}\|_{S^+} \lesssim t^{-1} \|\mathcal{R}[\widetilde{G}, \widetilde{G}, \widetilde{G}]\|_{S^+} \lesssim t^{-1} \|\widetilde{G}\|_{S^+} \|G_0\|_Z^2 + t^{-1+4\delta''} \|G_0\|_{S_+} \|G_0\|_S^3 \|G_0\|_Z^{-1} \ ,$$

thus using the inhomogeneous Gronwall's inequality, we gain the estimate of the S^+ norm in (3.4.13). □

Proposition 3.4.3. *If $A = \Pi_+ A$ and $B = \Pi_+ B$ solve (3.4.2) with $\mathcal{R}_+$ and satisfy*

$$\sup_{0 \le t \le T} \{ \|A(t)\|_Z + \|B(t)\|_Z \} \le \theta$$

and

$$\|A(0) - B(0)\|_{S^{(+)}} \le \delta$$

then, there holds that, for $0 \le t \le T$,

$$\|A(t) - B(t)\|_{S^{(+)}} \le \delta \mathrm{e}^{C\theta^2 t}. \tag{3.4.15}$$

Démonstration. By (3.4.2), $A - B$ satisfies

$$i\partial_t \Big(\widehat{A}_p(\xi) - \widehat{B}_p(\xi) \Big) = \mathcal{R}_+[\widehat{A}(\xi) - \widehat{B}(\xi), \widehat{A}(\xi), \widehat{A}(\xi)]_p \\ + \mathcal{R}_+[\widehat{B}(\xi), \widehat{A}(\xi) - \widehat{B}(\xi), \widehat{A}(\xi)]_p + \mathcal{R}_+[\widehat{B}(\xi), \widehat{B}(\xi), \widehat{A}(\xi) - \widehat{B}(\xi)]_p \ ,$$

then an application of Lemma 3.4.2 completes the proof. □

3.5 The main results

In this section, we will prove our main theorems. We will start with constructing a modified wave operator and gain the small data scattering as the theorem below.

3.5.1 Modified scattering

Given a small initial data in S^+, we may find a solution to our original system (3.1.1) by constructing a corresponding solution to the resonant system (3.1.10), which also leads to the global well-posedness of our wave guide Schrödinger equation with small data. In the other hand, the solution with small initial data admits some modified scattering property.

Theorem 3.5.1. *There exists $\varepsilon > 0$ such that if $U_0 \in S^+$ satisfies*

$$\|U_0\|_{S^+} \leq \varepsilon \ , \tag{3.5.1}$$

(1) If $\widetilde{G}$ is the solution of (3.1.10) with initial data U_0, then there exists a unique solution U of (3.1.1) such that $\mathrm{e}^{-it\mathcal{A}}U(t) \in C([0,\infty) : S)$ and

$$\|\mathrm{e}^{-it\mathcal{A}}U(t) - \widetilde{G}(\pi \ln t)\|_S \to 0 \ \text{ as } \ t \to +\infty \ .$$

(2) Conversely, consider the corresponding solution U of (3.1.1) with initial data U_0 satisfying (3.5.1), if ε is small enough, then there exists a solution $\widetilde{G}$ of (3.1.10), such that

$$\|\mathrm{e}^{-it\mathcal{A}}U(t) - \widetilde{G}(\pi \ln t)\|_S \to 0 \qquad \text{as } t \to +\infty \ . \tag{3.5.2}$$

Démonstration. Let us begin with (1). Set

$$G(t) = \widetilde{G}(\pi \ln t),\ K(t) = \mathrm{e}^{-it\mathcal{A}}U(t) - G(t)$$

and define a mapping

$$\Phi(K)(t) = -i\int_t^\infty \left\{\mathcal{N}^\sigma[K+G, K+G, K+G] - \frac{\pi}{\sigma}\mathcal{R}[G(\sigma), G(\sigma), G(\sigma)]\right\} d\sigma \ .$$

The main idea is to find a fixed point for Φ in a suitable space. Define

$$\begin{aligned} \mathfrak{A} :=& \{K \in C^1([1,\infty) : S) \ : \ \|K\|_{\mathfrak{A}} < \infty\} \\ \|K\|_{\mathfrak{A}} :=& \sup_{t>1} \left\{(1+|t|)^\delta \|K(t)\|_S + (1+|t|)^{2\delta}\|K(t)\|_Z + (1+|t|)^{1-\delta}\|\partial_t K(t)\|_S\right\} \ . \end{aligned}$$

We claim that if ε is sufficiently small, there exists ε_1 such that Φ defines a contraction on the complete metric space $\{K \in \mathfrak{A} \ : \ \|K\|_{\mathfrak{A}} \leq \varepsilon_1\}$. As in [26, Theorem 5.1], we decompose

$$\mathcal{N}^t[K+G, K+G, K+G] - \frac{\pi}{t}\mathcal{R}[G,G,G] = \mathcal{E}^t[G,G,G] + \mathcal{L}^t[K,G] + \mathcal{Q}^t[K,G] \tag{3.5.3}$$

where

$$\begin{cases} \mathcal{E}^t[G,G,G] := \mathcal{N}^t[G,G,G] - \frac{\pi}{t}\mathcal{R}[G,G,G] \ , \\ \quad \mathcal{L}^t[K,G] := \mathcal{N}^t[G,G,K] + \mathcal{N}^t[K,G,G] + \mathcal{N}^t[G,K,G] \ , \\ \quad \mathcal{Q}^t[K,G] := \mathcal{N}^t[K,K,G] + \mathcal{N}^t[G,K,K] + \mathcal{N}^t[K,G,K] + \mathcal{N}^t[K,K,K] \ . \end{cases}$$

For $K \in \mathfrak{A}$, we have

$$(1+|t|)^{2\delta}\|K(t)\|_Z + (1+|t|)^\delta\|K(t)\|_S + (1+|t|)^{1-\delta}\|\partial_t K(t)\|_S \lesssim \varepsilon_1 \ , \tag{3.5.4}$$

taking $\varepsilon \lesssim \delta^{1/2}$, by Proposition 3.4.2, we have

$$\begin{aligned} \|G(t)\|_{S^+} + (1+|t|)\|\partial_t G(t)\|_{S^+} \lesssim \varepsilon(1+|t|)^{\delta/100} \ , \\ \|G(t)\|_Z \lesssim \varepsilon \ . \end{aligned} \tag{3.5.5}$$

To show our claim, it suffices to show that the quantities below are small with $K, K_1, K_2 \in \mathfrak{A}$,

$$\| \int_t^\infty \mathcal{E}^\sigma[G,G,G] d\sigma \|_{\mathfrak{A}} \lesssim \varepsilon^3 \ , \tag{3.5.6}$$

$$\| \int_t^\infty \mathcal{L}^\sigma[K,G] d\sigma \|_{\mathfrak{A}} \lesssim \varepsilon^2 \|K\|_{\mathfrak{A}} \ , \tag{3.5.7}$$

$$\| \int_t^\infty \mathcal{Q}^\sigma[K,G] d\sigma \|_{\mathfrak{A}} \lesssim \varepsilon \|K\|_{\mathfrak{A}}^2 \ , \tag{3.5.8}$$

$$\| \int_t^\infty \left\{ \mathcal{Q}^\sigma[K_1,G] - \mathcal{Q}^\sigma[K_2,G] \right\} d\sigma \|_{\mathfrak{A}} \lesssim \varepsilon \varepsilon_1 \|K_1 - K_2\|_{\mathfrak{A}} \ . \tag{3.5.9}$$

Proof of (3.5.6)**.** Because of the definition of $\mathcal{E}^t$, we can easily gain for $t > 1$,

$$\begin{aligned} \|\mathcal{E}^t[G,G,G]\|_S &= \|\mathcal{N}^t[G,G,G] - \frac{1}{t}\mathcal{R}[G,G,G]\|_S \\ &\le \|\mathcal{N}^t[G,G,G]\|_S + \frac{1}{t}\|\mathcal{R}[G,G,G]\|_S \ . \end{aligned} \tag{3.5.10}$$

Using (3.2.18),

$$\|\mathcal{N}^t[G,G,G]\|_S \le t^{-1}\|G\|_S^3 \le t^{-1+\delta}\varepsilon^3 \ ,$$

while by (3.4.10),

$$\|\mathcal{R}^t[G,G,G]\|_S \le \|G\|_S^2 \|G\|_Z \le t^{\delta}\varepsilon^3 \ ,$$

then

$$\|\mathcal{E}^t[G,G,G]\|_S \le t^{-1+\delta}\varepsilon^3 \ ,$$

this controls the time derivative in the $\mathfrak{A}$ norm,

$$t^{1-\delta} \left\| \partial_t \Big(\int_t^\infty \mathcal{E}^\sigma[G,G,G] d\sigma \Big) \right\|_S \le \varepsilon^3 \ .$$

By (3.5.5), we have $\|G\|_{X_T^{(+)}} \le \varepsilon$ for any $T > 1$, so the other two terms of the $\mathfrak{A}$ norm, $\| \int_t^\infty \mathcal{E}^\sigma[G,G,G] d\sigma \|_S$ and $\| \int_t^\infty \mathcal{E}^\sigma[G,G,G] d\sigma \|_Z$ can be deduced by the estimates in Proposition 3.3.1.

Proof of (3.5.7)**.** We estimate the norm $\| \int_t^\infty \mathcal{N}^\sigma[G,G,K] d\sigma \|_{\mathfrak{A}}$ for example.

As in the proof of (3.5.6), using (3.2.18) and (3.5.5), we have the following estimate which controls the time derivative in the $\mathfrak{A}$ norm,

$$\|\mathcal{N}^t[G,G,K]\|_S \lesssim t^{-1}\|G\|_S^2 \|K\|_S \lesssim t^{-1+\delta}\varepsilon^2 \|K\|_{\mathfrak{A}} \ .$$

For the other two term in the $\mathfrak{A}$ norm, we shall reproduce the decomposition as in the proof of Proposition 3.3.1 on $\mathcal{N}^t[G,G,K]$. Using Lemma 3.3.2 and Lemma 3.3.4, it only remains to show that

$$\|\mathcal{R}[G,G,K]\|_Z \lesssim (1+|t|)^{-2\delta}\varepsilon^2\varepsilon_1 \ , \tag{3.5.11}$$

$$\|\mathcal{N}_0^t[G,G,K] - \frac{\pi}{t}\mathcal{R}[G,G,K]\|_Z \lesssim (1+|t|)^{-1-2\delta}\varepsilon^2\varepsilon_1 \ , \tag{3.5.12}$$

$$\|\mathcal{N}_0^t[G,G,K]\|_S \lesssim (1+|t|)^{-1-\delta}\varepsilon^2\varepsilon_1 \ . \tag{3.5.13}$$

The first estimate follows from (3.4.9),

$$\|\mathcal{R}[G,G,K]\|_Z \lesssim \|G\|_Z^2\|K\|_Z \lesssim (1+|t|)^{-2\delta}\varepsilon^2\varepsilon_1 \ .$$

The second estimate follows from (3.3.42),

$$\|\mathcal{N}_0^t[G,G,K] - \frac{\pi}{t}\mathcal{R}[G,G,K]\|_Z \lesssim (1+|t|)^{-1-20\delta}\|G\|_S^2\|K\|_S \lesssim (1+|t|)^{-1-2\delta}\varepsilon^2\varepsilon_1 \ .$$

For the third estimate, we use (3.3.40) to get

$$\begin{aligned}(1+|t|)\left\{\|\mathcal{N}_0^t[G,G,K]\|_S + \|\mathcal{N}_0^t[G,K,G]\|_S\right\} &\lesssim \|G\|_{\widetilde{Z}_t}^2\|K\|_S + \|G\|_{\widetilde{Z}_t}\|K\|_{\widetilde{Z}_t}\|G\|_S \\ &\lesssim \varepsilon^2\varepsilon_1(1+|t|)^{-\delta}.\end{aligned}$$

Proof of (3.5.8)**.** The proof of (3.5.8) is similar to the proof of (3.5.7).

Proof of (3.5.9)**.** We may rewrite

$$\begin{aligned}\mathcal{N}^t[K_1,K_1,G] - \mathcal{N}^t[K_2,K_2,G] &= \mathcal{N}^t[K_1,K_1,G] - \mathcal{N}^t[K_1,K_2,G] + \mathcal{N}^t[K_1,K_1,G] \\ &\qquad - \mathcal{N}^t[K_2,K_2,G] \\ &= \mathcal{N}^t[K_1,K_1-K_2,G] + \mathcal{N}^t[K_1-K_2,K_2,G] \ ,\end{aligned}$$

we take similar decompositions on the terms $\mathcal{N}^t[K_1,G,K_1] - \mathcal{N}^t[K_2,G,K_2]$, $\mathcal{N}^t[G,K_1,K_1] - \mathcal{N}^t[G,K_2,K_2]$ and $\mathcal{N}^t[K_1,K_1,K_1] - \mathcal{N}^t[K_2,K_2,K_2]$. Similar strategy we used to prove (3.5.7) can be applied to obtain the estimate on the norm $\|\mathcal{N}^t[F_1,F_2,F_3]\|_{\mathfrak{A}}$ with one of these F_j be K_1-K_2 while the other two functions belong to $\{K_1,K_2,G\}$. The proof of the first part is complete.

Let us turn to (2), we will prove it in two steps.

Step 1 : Global existence and bounds. Let $U_0 \in S^+$, $\|U_0\|_{S^+} \le \varepsilon$ with ε small enough. The local existence is classical via its integral equation. We denote $F(t) := e^{-it\mathcal{A}}U(t)$, then (3.1.1) can be rewritten as

$$i\partial_t F = \mathcal{N}^t[F,F,F] \ . \tag{3.5.14}$$

$$F(t) = U_0 - i\int_0^t \mathcal{N}^\sigma[F,F,F]d\sigma \ . \tag{3.5.15}$$

By the estimate (3.2.18), we have

$$\|\mathcal{N}^t[F,F,F]\|_{S^+} \lesssim (1+|t|)^{-1}\|F\|_{S^+}^3 \ .$$

This allows us to use a fixed point argument on a small time interval $[0,T]$, and $t \mapsto \|F(t)\|_{S^+}$ is C^1. We claim that

$$\|F\|_{X_T^+} \le \|U_0\|_{S^+} + C\|F\|_{X_T^+}^3 \tag{3.5.16}$$

for all $T > 0$ and all U solving (3.1.1) such that $\|F\|_{X_T^+} \leq \sqrt{\varepsilon}$. Then by a bootstrap argument, we gain the global existence and the solution satisfies for all $T > 0$

$$\|FU(t)\|_{X_T^+} \leq 2\varepsilon \; . \tag{3.5.17}$$

Let us begin the proof of our claim (3.5.16). Recall the definition of the X_T^+ norm (3.2.11), we have to consider the S and S^+ norm of F and $\partial_t F$ and also the Z norm of F.

It is easy to deduce from the equation on F that $\|\partial_t F\|_{S^{(+)}} = \|\mathcal{N}^t[F,F,F]\|_{S^{(+)}}$. Thanks to (3.2.18), we have

$$\begin{aligned}
&\|\partial_t F\|_S = \|\mathcal{N}^t[F,F,F]\|_S \leq (1+|t|)^{-1}\|F\|_S^3 \; , \\
&\|\partial_t F\|_{S^+} = \|\mathcal{N}^t[F,F,F]\|_{S^+} \leq (1+|t|)^{-1}\|F\|_S^2\|F\|_{S^+} \; ,
\end{aligned}$$

thus

$$(1+|t|)^{1-3\delta}\|\partial_t F\|_S \leq \big((1+|t|)^{-\delta}\|F\|_S\big)^3 + \leq \|F\|_{X_T^+}^3 \; , \tag{3.5.18}$$

$$(1+|t|)^{1-7\delta}\|\partial_t F\|_{S^+} \leq \big((1+|t|)^{-\delta}\|F\|_S\big)^2(1+|t|)^{-5\delta}\|F\|_{S^+} \leq \|F\|_{X_T^+}^3 \; . \tag{3.5.19}$$

We now turn to estimate $\|F\|_Z$, by the decomposition result of $\mathcal{N}^t$ in Proposition 3.3.1, and notice that R defined as (3.4.4) is self-adjoint on ℓ_p^2 and that there is a cancellation

$$\langle i\mathcal{F}\mathcal{R}[F,F,F](\xi),\ \mathcal{F}F(\xi)\rangle_{h_p^\sigma,h_p^\sigma} = 0 \; .$$

So we will study the $\|F\|_{Y^\sigma}$ with $\sigma > 1$ where Y^σ is defined in (3.2.8), then to control the Z norm.

$$\begin{aligned}
\frac{d}{ds}\frac{1}{2}\|\widehat{F}_p(\xi,s)\|_{h_p^\sigma}^2 &= \Big\langle \mathcal{F}\mathcal{N}^t[F,F,F](\xi,s),\ \widehat{F}_p(\xi,s)\Big\rangle_{h_p^\sigma,h_p^\sigma} \\
&= \langle \widehat{\mathcal{E}}_1(\xi,p,s), \widehat{F}_p(\xi,s)\rangle_{h_p^\sigma\times h_p^\sigma} + \langle \partial_s\widehat{\mathcal{E}}_3(\xi,p,s), \widehat{F}_p(\xi,s)\rangle_{h_p^\sigma\times h_p^\sigma} \; .
\end{aligned} \tag{3.5.20}$$

Thus multiplying with $(1+|\xi|^2)$, using the estimates of $\|\mathcal{E}_j\|_{Y^\sigma}$ in Proposition 3.3.1, then we have for any ξ, we have

$$(1+|\xi|^2)\Big|\int_0^t \langle \widehat{\mathcal{E}}_1(\xi,p,s), \widehat{F}_p(\xi,s)\rangle_{h_p^\sigma\times h_p^\sigma}\Big| \lesssim \|F\|_{X_T^+}^3 \int_0^t (1+|s|)^{-1-\delta}ds \cdot \sup_{[0,t]}\|F(s)\|_{Y^\sigma} \; ,$$

while

$$\begin{aligned}
&[1+|\xi|^2]\left|\int_0^t \langle \partial_t\widehat{\mathcal{E}}_3(\xi,p,s), \widehat{F}_p(\xi,s)\rangle_{h^\sigma\times h_p^\sigma} ds\right| \leq [1+|\xi|^2]\left|\langle \widehat{\mathcal{E}}_3(\xi,p,t), \widehat{F}_p(\xi,t)\rangle_{h^\sigma\times h_p^\sigma}\right| \\
&\qquad + [1+|\xi|^2]\left|\langle \widehat{\mathcal{E}}_3(\xi,p,0), \widehat{F}_p(\xi,0)\rangle_{h^\sigma\times h_p^\sigma}\right| + [1+|\xi|^2]\left|\int_0^t \langle \widehat{\mathcal{E}}_3(\xi,p,s), \partial_t\widehat{F}_p(\xi,s)\rangle_{h^\sigma\times h_p^\sigma}\right| \\
&\lesssim \|F\|_{X_T^+}^3 \cdot \sup_{[0,t]}\|F(s)\|_{Y^\sigma} + \|F\|_{X_T^+}^6 \; .
\end{aligned}$$

Combining the above estimates, we have

$$\|F(t)\|_Z \leq \|F(t)\|_{Y^\sigma} + C\|F\|_{X_T^+}^3 \; . \tag{3.5.21}$$

For the norm $\|F(t)\|_{S^{(+)}}$, when $0 \le t \le 1$,

$$\begin{aligned}\|F\|_S \le \|F\|_{S^+} &\le \|U_0\|_{S^+} + \|F(t) - F(0)\|_{S^+} \\ &\le \|U_0\|_{S^+} + \sup_{0 \le t \le 1} \|\partial_t F\|_{S^+} \\ &\le \|F\|^3_{X_T^+} .\end{aligned}$$

While $1 \le t \le T$, using Proposition 3.3.1, we have

$$\begin{aligned}\|F(t) - F(1)\|_{S^{(+)}} &\le \left\| \int_1^t \mathcal{N}^\sigma[F, F, F] d\sigma \right\|_{S^{(+)}} \\ &\le \left\| \int_1^t \mathcal{R}[F, F, F] d\sigma/\sigma \right\|_{S^{(+)}} + \left\| \int_1^t \big(\mathcal{E}_1(\sigma) + \mathcal{E}_2(\sigma)\big) d\sigma \right\|_{S^{(+)}} ,\end{aligned}$$

then using (3.3.44) and the, we have

$$\begin{aligned}\left\| \int_1^t \mathcal{R}[F, F, F] d\sigma/\sigma \right\|_S &\lesssim \int_1^t \sigma^{-1} \|F\|^2_{\widetilde{Z}_t} \|F\|_S d\sigma && (3.5.22) \\ &\lesssim \int_1^t \sigma^{-1+\delta} d\sigma \|F\|^3_{X_T^+} \le t^\delta \|F\|^3_{X_T^+} , && (3.5.23)\end{aligned}$$

while by (3.3.45),

$$\begin{aligned}\left\| \int_1^t \mathcal{R}[F, F, F] d\sigma/\sigma \right\|_{S^+} &\le \int_1^t \left(\sigma^{-1} \|F\|^2_{\widetilde{Z}_t} \|F\|_{S^+} + \sigma^{-1+2\delta} \|F\|_{\widetilde{Z}_t} \|F\|^2_S \right) d\sigma && (3.5.24) \\ &\le \int_1^t \sigma^{-1+5\delta} d\sigma \|F\|^3_{X_T^+} \le t^{5\delta} \|F\|^3_{X_T^+} , && (3.5.25)\end{aligned}$$

together with the estimates in Proposition 3.3.1,

$$\|F(t) - F(1)\|_S \le (1 + |t|)^\delta \|F\|^3_{X_T^+} , \tag{3.5.26}$$

$$\|F(t) - F(1)\|_{S^+} \le (1 + |t|)^{5\delta} \|F\|^3_{X_T^+} . \tag{3.5.27}$$

Hence, we finally gain

$$(1 + |t|)^{-\delta} \|F\|_S + (1 + |t|)^{-5\delta} \|F\|_{S^+} \le \|F\|_{X_T^+} . \tag{3.5.28}$$

Our priori estimate (3.5.16) comes out from (3.5.18), (3.5.19), (3.5.21) and (3.5.28).

Step 2 : Asymptotic behavior. Define $T_n = e^{n/\pi}$ and $G_n(t) = \widetilde{G}_n(\pi \ln t)$, where $\widetilde{G}_n$ solves (3.1.10) with Cauchy data such that $\widetilde{G}_n(n) = G_n(T_n) = F(T_n)$. We claim that for all $t \ge T_n$,

$$\|G_n(t)\|_Z + (1 + |t|)^{-\delta} \|G_n(t)\|_S + (1 + |t|)^{-5\delta} \|G_n(t)\|_{S^+} + (1 + |t|)^{1-\delta} \|\partial_t G_n(t)\|_S \lesssim \varepsilon \tag{3.5.29}$$

uniformly in $n \ge 0$. Indeed, first we get from the global bounds result (3.5.17) that uniformly in n,

$$\|G_n(t)\|_Z = \|\widetilde{G}_n(\pi \ln t)\|_Z = \|\widetilde{G}_n(n)\|_Z = \|F(T_n)\|_Z \lesssim \varepsilon ,$$

$$\|G_n(T_n)\|_S \lesssim \varepsilon T_n^{\delta} \ ,$$

and by (3.4.10),

$$\|\partial_t G_n(s)\|_S \lesssim s^{-1}\|G_n\|_Z^2\|G_n(s)\|_S \lesssim \varepsilon^2 s^{-1}\|G_n(s)\|_S \ . \tag{3.5.30}$$

An application of Gronwall's lemma gives, for ε small enough,

$$\|G_n(s)\|_S \lesssim \varepsilon s^{\delta}, \quad s \geq T_n$$

which, combined with (3.5.30), provides control of the second and last term in (3.5.29). We can estimate the S^+ norm similarly, using (3.3.45),

$$\|\partial_t G_n(s)\|_{S^+} \lesssim s^{-1}\varepsilon^2\|G_n(s)\|_{S^+} + \varepsilon^3 s^{-1+4\delta}, \qquad \|G_n(T_n)\|_{S^+} \lesssim \varepsilon T_n^{5\delta}.$$

This concludes the proof of (3.5.29).

Since

$$\begin{aligned} i\partial_t F &= \mathcal{N}^t[F,F,F] \ , \\ i\partial_t G_n &= \frac{t}{\pi}\mathcal{R}[G_n,G_n,G_n] \ , \end{aligned}$$

and

$$F(T_n) = G_n(T_n) \ ,$$

then

$$\begin{aligned} F(t) - G_n(t) &= i\int_{T_n}^t \Big(\mathcal{N}^\sigma[F,F,F] - \frac{\sigma}{\pi}\mathcal{R}[G_n,G_n,G_n]\Big)d\sigma \\ &= i\int_{T_n}^t \mathcal{E}^\sigma[F,F,F]d\sigma + i\int_{T_n}^t \frac{\sigma}{\pi}\Big(\mathcal{R}[F,F,F] - \mathcal{R}[G_n,G_n,G_n]\Big)d\sigma \ . \end{aligned}$$

Using the estimates in Proposition 3.3.1, we gain for $t > T_n$,

$$\begin{aligned} \|F - G_n\|_Z &\lesssim \varepsilon^3 T_n^{-2\delta} + \int_{T_n}^t \big(\|F\|_Z^2 + \|G_n\|_Z^2\big)\|F - G_n\|_Z \frac{d\sigma}{\sigma} \\ &\lesssim \varepsilon^3 T_n^{-2\delta} + \varepsilon^2\int_{T_n}^t \|F - G_n\|_Z \frac{d\sigma}{\sigma} \ , \end{aligned}$$

we may then deduce by Gronwall,

$$\|F - G_n\|_Z \lesssim \varepsilon^3 T_n^{-2\delta} \text{ for } T_n \leq t \leq T_{n+4} \ .$$

We may deduce the estimate on $\|F - G_n\|_S$ similarly and

$$\begin{aligned} \|F - G_n\|_S \lesssim{}& \varepsilon^3 T_n^{-2\delta} + \int_{T_n}^t \big(\|F\|_Z^2 + \|G_n\|_Z^2\big)\|F - G_n\|_S \frac{d\sigma}{\sigma} \\ &+ \int_{T_n}^t \big(\|F\|_Z + \|G_n\|_Z\big)\|F - G_n\|_Z\big(\|F\|_S + \|G_n\|_S\big)\frac{d\sigma}{\sigma} \\ \lesssim{}& \varepsilon^3 T_n^{-\delta} + \varepsilon^2\int_{T_n}^t \|F - G_n\|_S \frac{d\sigma}{\sigma} \ , \end{aligned}$$

Again, we use Gronwall's inequality and get

$$\sup_{T_n \leq t \leq T_{n+4}} \|F(t) - G_n(t)\|_S \lesssim \varepsilon^3 T_n^{-\delta} \ . \tag{3.5.31}$$

Therefore,

$$\|\widetilde{G}_{n+1}(n+1) - \widetilde{G}_n(n+1)\|_S = \|F(T_{n+1}) - G_n(T_{n+1})\|_S \lesssim \varepsilon^3 T_n^{-\delta} \ , \tag{3.5.32}$$

and thus by Lemma 3.4.3, we gain

$$\|\widetilde{G}_{n+1}(0) - \widetilde{G}_n(0)\|_S \lesssim \varepsilon^3 \mathrm{e}^{-n\delta/2} \ .$$

We see that $\{\widetilde{G}_n(0)\}_n$ is a Cauchy sequence in S and therefore converges to an element $G_{0,\infty} \in S$ which satisfies that

$$\|G_{0,\infty}\|_Z \lesssim \varepsilon, \qquad \|\widetilde{G}_n(0) - G_{0,\infty}\|_S \lesssim \varepsilon^3 \mathrm{e}^{-n\delta/2}.$$

By Proposition 3.4.3,

$$\sup_{[0,T_{n+2}]} \|G_\infty(t) - G_n(t)\|_S \lesssim \varepsilon^3 \mathrm{e}^{-n\delta/4}$$

where $G_\infty(t) = \widetilde{G}_\infty(\pi \ln t)$ with $\widetilde{G}_\infty$ the solution of (3.1.10) with initial data $\widetilde{G}_\infty(0) = G_{0,\infty}$. Now we have

$$\begin{aligned}\sup_{T_n \leq t \leq T_{n+1}} \|G_\infty(t) - F(t)\|_S &\leq \sup_{T_n \leq t \leq T_{n+1}} \|G_\infty(t) - G_n(t)\|_S + \sup_{T_n \leq t \leq T_{n+1}} \|G_n(t) - F(t)\|_S \\ &\lesssim \varepsilon^3 e^{-n\delta/4}.\end{aligned}$$

This finishes the proof. □

3.5.2 Large time Sobolev unboundedness

We will firstly study the dynamics of the resonant system (3.1.10), then we apply the modified scattering results above to gain the large time behavior of the wave guide Schrödinger equation. The following strategy allows us to transfer informations from a global solution $a(t)$ of (3.4.4) to a solution of (3.4.2), all we need to do is to take an initial datum of the form

$$G_0(x, y) = \check{\varphi}(x) g(y), \ \varphi \in \mathcal{S}(\mathbb{R}) \ ,$$

where $g_p = a_p(0)$, and $\check{\varphi}(x)$ is the inverse Fourier transform of φ. The solution $G(t)$ to (3.4.2) with initial data G_0 as above is given in Fourier space by

$$\widehat{G}_p(t, \xi) = \varphi(\xi) a_p(\varphi(\xi)^2 t) \ . \tag{3.5.33}$$

In particular, if $\varphi = 1$ on an open interval I, then $\widehat{G}_p(t, \xi) = a_p(t)$ for all $t \in \mathbb{R}$ and $\xi \in I$. For $\xi \in I$, the resonant system turns out to be the cubic Szegő equation.

Let us recall the infinite cascade result for the cubic Szegő equation.

Theorem 3.5.2. *[17] For any $v_0 \in C_+^\infty := \cap_s H^s$, for any M, for any $r > \frac{1}{2}$, there exists a sequence (v_0^n) of elements of C_+^∞ tending to v_0 in C_+^∞ and a sequence of time t_n, $|t_n|$ tending to ∞, such that the corresponding solution v_n of the cubic Szegő equation*

$$i\partial_t v = \Pi_+(|v|^2 v) \ , \ v(0) = v_0^n \ , \tag{3.5.34}$$

satisfies

$$\frac{\|v^n(t^n)\|_{H^r}}{|t_n|^M} \to \infty \ , \ n \to \infty \ . \tag{3.5.35}$$

Assumption 3.5.1. *$G_0 \in S^+$ with $\|G_0\|_{S^+}$ small, $G_0(y) = -G_0(y+\pi)$, and there exists some non empty open set $I \neq \emptyset$, such that $G_0(\xi) = v_0$ $\forall \xi \in I$, while the corresponding solution of the cubic Szegő equation (3.5.34) with v_0 as the initial data admits an unbounded trajectory as described in Theorem 3.5.2.*

Let G be a solution to

$$\begin{cases} i\partial_t G = \mathcal{R}[G,G,G] \\ G(0) = G_0 \end{cases} \tag{3.5.36}$$

with G_0 satisfies Assumption 3.5.1, then

$$\|G(t)\|_{L_x^2 H_y^s} \geq \Big(\int_I \|\widehat{G}(t,\xi)\|_{H_y^s}^2 d\xi \Big)^{1/2} = |I|^{1/2} \|v(t)\|_{H_y^s} \to \infty \ , \tag{3.5.37}$$

for any $s > 1/2$.

By Theorem 3.5.1, for the solutions to the resonant system G as above, there exists solutions to the wave guide Schrödinger equation (3.1.1), such that (3.5.1) holds. Then the large time behavior of $G(t)$ (3.5.37) leads to the large time unbounded Sobolev trajectories of solutions to the equation (3.1.1).

Corollary 3.5.1. *Given $N \geq 13$, then for any $\varepsilon > 0$, there exists $U_0 \in S^+$ with $\|U_0\|_{S^+} \leq \varepsilon$ such that the corresponding solution to (3.1.1) satisfies*

$$\limsup_{t \to \infty} \frac{\|U(t)\|_{L_x^2 H_y^s}}{(1 + \log|t|)^M} = \infty \ , \quad \forall s > 1/2 \ , \ \forall M \ . \tag{3.5.38}$$

Remark 3.5.1. *As we announced in the introduction of this paper, the unbounded Sobolev norms in our theorem are just above the energy norm. Recall the results of upper bounds on the dispersive equations by Bourgain [3] and Staffilanic [47], the superior growth is polynomial in time. Here, the growth is as large as $(\log|t|)^M$ for any M for solutions with small initial data in S^+, which is almost optimal for the dispersive wave guide Schrödinger equation.*

Moreover, in view of Theorem 3.5.2 by Gérard and Grellier [17], we expect that there exist some Banach space B, such that the set

$$G := \left\{ U_0 \in B \ : \ \forall s > \frac{1}{2} \ , \forall M \in \mathbb{Z}_+ \ , \ \limsup_{|t| \to +\infty} \frac{\|U(t)\|_{H^s}}{(\log|t|)^M} = +\infty \right\}$$

is a dense G_δ subset of B. The difficulty comes from the gap between S and S^+ in the modified scattering argument, which already exists in the early results of Ozawa [39] and Hayashi–Naumkin–Shinomura–Tonegawa [32].

3.6 Appendix

We now turn to our basic lemma allowing to transform suitable $L^2_{x,y}$ bounds to bounds in terms of the $L^2_{x,y}$-based spaces S and S^+. We define an *LP-family* $\widetilde{Q} = \{\widetilde{Q}_A\}_A$ to be a family of operators (indexed by the dyadic integers) of the form

$$\widehat{\widetilde{Q}_1 f}(\xi) = \widetilde{\varphi}(\xi)\widehat{f}(\xi), \qquad \widehat{\widetilde{Q}_A f}(\xi) = \widetilde{\phi}(\frac{\xi}{A})\widehat{f}(\xi), \quad A \geq 2$$

for two smooth functions $\widetilde{\varphi}, \widetilde{\phi} \in C_c^\infty(\mathbb{R})$ with $\widetilde{\phi} \equiv 0$ in a neighborhood of 0.

We define the set of *admissible transformations* to be the family of operators $\{T_A\}$ where for any dyadic number A,

$$T_A = \lambda_A \widetilde{Q}_A, \qquad |\lambda_A| \leq 1$$

for some LP-family $\widetilde{Q}$.

If $F \in \mathcal{B}$, then for any admissible transformation family $T = \{T_A : A \text{ dyadic numbers }\}$, $\sum_A T_A F$ converges in $\mathcal{B}$. And this norm $\mathcal{B}$ is called admissible if

$$\|\sum_A T_A F\|_{\mathcal{B}} \lesssim \|F\|_{\mathcal{B}}. \tag{3.6.1}$$

Lemma 3.6.1. *Recall the definitions of the norms S, S^+, Z and $\widetilde{Z}_t$,*

$$\|F\|_S := \|F\|_{H^N_{x,y}} + \|xF\|_{L^2_{x,y}} , \quad \|F\|_{S^+} := \|F\|_S + \|(1-\partial_{xx})^4 F\|_S + \|xF\|_S ,$$
$$\|F\|_Z := \sup_{\xi\in\mathbb{R}} \left[1+|\xi|^2\right] \|\widehat{F}(\xi,\cdot)\|_{B^1} , \quad \|F\|_{\widetilde{Z}_t} := \|F\|_Z + (1+|t|)^{-\delta}\|F\|_S .$$

All these norms are admissible.

Démonstration. Due to the definition of admissible transformation, we may only deal with functions independent on y. Let us prove with the S norm for example. Indeed,

$$\begin{aligned}
\|\sum_A T_A f\|^2_{H^N} &= \int_{\mathbb{R}} \langle\xi\rangle^{2N} \Big|\lambda_1\widetilde{\varphi}(\xi)\widehat{f}(\xi) + \lambda_A\widetilde{\phi}(\frac{\xi}{A})\widehat{f}(\xi)\Big|^2 d\xi \\
&\leq \int_{\mathbb{R}} \Big(|\lambda_1|^2\widetilde{\varphi}(\xi) + |\lambda_A|^2\widetilde{\phi}(\frac{\xi}{A})\Big)\langle\xi\rangle^{2N}|\widehat{f}(\xi)|^2 d\xi \\
&\leq \|f\|^2_{H^N} ,
\end{aligned}$$

while

$$\begin{aligned}
\|x\sum_A T_A f\|^2_{L^2} &= \int_{\mathbb{R}} \Big|\partial_\xi\big(\lambda_1\widetilde{\varphi}(\xi)\widehat{f}(\xi) + \lambda_A\widetilde{\phi}(\frac{\xi}{A})\widehat{f}(\xi)\big)\Big|^2 d\xi \\
&\leq \int_{\mathbb{R}} \Big(|\lambda_1|^2\widetilde{\varphi}(\xi) + |\lambda_A|^2\widetilde{\phi}(\frac{\xi}{A})\Big)|\partial_\xi\widehat{f}(\xi)|^2 d\xi \\
&\quad + \int_{\mathbb{R}} \Big(|\lambda_1|^2\widetilde{\varphi}'(\xi) + \frac{|\lambda_A|^2}{A}\widetilde{\phi}'(\frac{\xi}{A})\Big)|\widehat{f}(\xi)|^2 d\xi \\
&\leq \|xf\|^2_{L^2} + \|f\|^2_{L^2} ,
\end{aligned}$$

thus

$$\|\sum_A T_A f\|_S \lesssim \|f\|_S \ .$$

□

Given a trilinear operator $\mathfrak{T}$ and a set Λ of 4-tuples of dyadic integers, we define an *admissible realization* of $\mathfrak{T}$ at Λ to be an operator of the form which converges in L^2,

$$\mathfrak{T}_\Lambda[F,G,H] = \sum_{(A,B,C,D)\in\Lambda} T_D \mathfrak{T}[T'_A F, T''_B G, T'''_C H] \tag{3.6.2}$$

for some admissible transformations T, T', T'', T'''.

Lemma 3.6.2. *Assume that a trilinear operator $\mathfrak{T}$ satisfies*

$$Z\mathfrak{T}[F,G,H] = \mathfrak{T}[ZF,G,H] + \mathfrak{T}[F,ZG,H] + \mathfrak{T}[F,G,ZH], \tag{3.6.3}$$

for $Z \in \{x, \partial_x, \partial_y\}$ and let Λ be a set of 4-tuples of dyadic integers. With the notation introduced above, assume also that for all admissible realizations of $\mathfrak{T}$ at Λ,

$$\|\mathfrak{T}_\Lambda[F^a,F^b,F^c]\|_{L^2} \le K \min_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{L^2}\|F^\beta\|_{\mathcal{B}}\|F^\gamma\|_{\mathcal{B}} \tag{3.6.4}$$

for some admissible norm $\mathcal{B}$ such that the Littlewood-Paley projectors $P_{\le M}$ (both in x and in y) are uniformly bounded on $\mathcal{B}$. Then, for all admissible realizations of $\mathfrak{T}$ at Λ,

$$\|\mathfrak{T}_\Lambda[F^a,F^b,F^c]\|_S \lesssim K \max_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_S\|F^\beta\|_{\mathcal{B}}\|F^\gamma\|_{\mathcal{B}} \ . \tag{3.6.5}$$

Assume in addition that, for $Y \in \{x, (1-\partial_{xx})^4\}$,

$$\|YF\|_{\mathcal{B}} \lesssim \theta_1\|F\|_{S^+} + \theta_2\|F\|_S \ , \tag{3.6.6}$$

then for all admissible realizations of $\mathfrak{T}$ at Λ,

$$\begin{aligned}\|\mathfrak{T}_\Lambda[F^a,F^b,F^c]\|_{S^+} \lesssim K &\max_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_{S^+}\big(\|F^\beta\|_{\mathcal{B}} + \theta_1\|F^\beta\|_S\big)\|F^\gamma\|_{\mathcal{B}} \\ &+ \theta_2 K \max_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|_S\|F^\beta\|_S\|F^\gamma\|_{\mathcal{B}} \ .\end{aligned} \tag{3.6.7}$$

Démonstration. Let us start with (3.6.5).
1. The weighted component of S norm. By rewriting $xT_A = [x,T_A] + T_A x$ and using (3.6.3), we have

$$\begin{aligned} x\mathfrak{T}_\Lambda[F^a,F^b,F^c] &= \sum_{(A,B,C,D)\in\Lambda} xT_D\mathfrak{T}[T'_AF^a, T''_BF^b, T'''_CF^c] \\ &= \sum_{(A,B,C,D)\in\Lambda} [x,T_D]\mathfrak{T}[T'_AF^a, T''_BF^b, T'''_CF^c] + \sum_{(A,B,C,D)\in\Lambda} T_D\mathfrak{T}[[x,T'_A]F^a, T''_BF^b, T'''_CF^c] \\ &+ \sum_{(A,B,C,D)\in\Lambda} T_D\mathfrak{T}[T'_AF^a, [x,T''_B]F^b, T'''_CF^c] + \sum_{(A,B,C,D)\in\Lambda} T_D\mathfrak{T}[T'_AF^a, T''_BF^b, [x,T'''_C]F^c] \\ &+ \mathfrak{T}_\Lambda[xF^a,F^b,F^c] + \mathfrak{T}_\Lambda[F^a,xF^b,F^c] + \mathfrak{T}_\Lambda[F^a,F^b,xF^c] \ . \end{aligned} \tag{3.6.8}$$

By simple calculation, we have

$$[x, Q_A] = A^{-1} Q'_A \ .$$

We notice that if Q_A is an LP-family, Q'_A is also an LP-family, then $[x, T_A]$ is also an admissible transformation. Thus, we may consider $x\mathfrak{T}_\Lambda[F^a, F^b, F^c]$ as the following summation

$$\mathfrak{T}_\Lambda[F^a, F^b, F^c] + \mathfrak{T}_\Lambda[xF^a, F^b, F^c] + \mathfrak{T}_\Lambda[F^a, xF^b, F^c] + \mathfrak{T}_\Lambda[F^a, F^b, xF^c] \ , \tag{3.6.9}$$

then $\|x\mathfrak{T}_\Lambda[F^a, F^b, F^c]\|_{L^2}$ follows from (3.6.4).

2. The H^N component of S norm. We will use the equivalent definition of H^N norm,

$$\|F\|^2_{H^N} := \sum_{M \text{ dyadic}} M^{2N} \|P_M F\|^2_{L^2} \ ,$$

with P_M as the Littlewood-Paley projections on $\mathbb{R} \times \mathbb{T}$ defined in Section 2. Then, we may decompose

$$P_M \mathfrak{T}_\Lambda[F^a, F^b, F^c] = P_M \mathfrak{T}_{\Lambda,low}[F^a, F^b, F^c] + P_M \mathfrak{T}_{\Lambda,high}[F^a, F^b, F^c] \ ,$$

with$\mathfrak{T}_{\Lambda,low}[F^a, F^b, F^c] = \mathfrak{T}_\Lambda[P_{\leq M}F^a, P_{\leq M}F^b, P_{\leq M}F^c]$.

We have firstly

$$\sum_{M \text{ dyadic}} M^{2N} \|P_M \mathfrak{T}_{\Lambda,high}[F^a, F^b, F^c]\|^2_{L^2} \lesssim K^2 \max_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|^2_{H^N} \|F^\beta\|^2_{\mathcal{B}} \|F^\gamma\|^2_{\mathcal{B}} \ , \tag{3.6.10}$$

since

$$\begin{aligned}\sum_M |M|^{2N} \|P_M \mathfrak{T}_\Lambda[P_{\geq 2M}F^a, F^b, F^c]\|^2_{L^2} &\leq K^2 \sum_M |M|^{2N} \|P_{\geq 2M}F^a\|^2_{L^2} \|F^b\|^2_{\mathcal{B}} \|F^c\|^2_{\mathcal{B}} \\ &\lesssim K^2 \|F^a\|^2_{H^N} \|F^b\|^2_{\mathcal{B}} \|F^c\|^2_{\mathcal{B}} \ .\end{aligned} \tag{3.6.11}$$

Let $Z \in \{\partial_x, \partial_y\}$, we can bound the contribution of $\mathfrak{T}_{\Lambda,low}$ as below

$$\begin{aligned}&M^N \|P_M \mathfrak{T}_{\Lambda,low}\|_{L^2} \\ &\quad \lesssim M^{-N} \|Z^{2N} P_M \mathfrak{T}_{\Lambda,low}[P_{\leq M}F^a, P_{\leq M}F^b, P_{\leq M}F^c]\|_{L^2} \\ &\quad = M^{-N} \| \sum_{\alpha+\beta+\gamma \leq 2N} \sum_{M_1,M_2,M_3 \leq M} P_M \mathfrak{T}_{\Lambda,low}[Z^\alpha P_{M_1}F^a, Z^\beta P_{M_2}F^b, Z^\gamma P_{M_3}F^c]\|_{L^2} \ .\end{aligned} \tag{3.6.12}$$

Without loss of generality, we assume $M_1 = \max(M_1, M_2, M_3) \leq M$, then

$$\begin{aligned}M^N \|P_M \mathfrak{T}_{\Lambda,low}\|_{L^2} &\lesssim \sum_{M_1 \leq M} M^{-N} M_1^{2N} \sum_{M_2,M_3 \leq M_1} \|\mathfrak{T}_{\Lambda,low}[P_{M_1}F^a, P_{M_2}F^b, P_{M_3}F^c]\|_{L^2} \\ &\lesssim K \sum_{M_1 \leq M} (\frac{M_1}{M})^{-N} M_1^N \|P_{M_1}F^a\|_{L^2} \|F^b\|_{\mathcal{B}} \|F^c\|_{\mathcal{B}} \ ,\end{aligned} \tag{3.6.13}$$

the above sum is in ℓ^2_M by Schur test, then

$$\sum_{M \text{ dyadic}} M^{2N}\|P_M\mathfrak{T}_{\Lambda,low}[F^a,F^b,F^c]\|^2_{L^2} \lesssim K^2 \max_{\{\alpha,\beta,\gamma\}=\{a,b,c\}} \|F^\alpha\|^2_{H^N}\|F^\beta\|^2_{\mathcal{B}}\|F^\gamma\|^2_{\mathcal{B}}\ . \tag{3.6.14}$$

Therefore we bound the H^N component of S norm, which completes the estimate (3.6.5).

Now, we turn to prove the estimate (3.6.7), due to the definition of S^+ norm, we only need to bound $\|x\mathfrak{T}_\Lambda\|_S$ and $\|(1-\partial_{xx})^4\mathfrak{T}_\Lambda\|_S$. From (3.6.9) and (3.6.5), we gain directly

$$\begin{aligned}&\|x\mathfrak{T}_\Lambda[F^a,F^b,F^c]\|_S\\ &\quad\lesssim \max_{\{\alpha,\beta,\gamma\}=\{a,b,c\}}\|F^\alpha\|_{S^+}\|F^\beta\|_{\mathcal{B}}\|F^\gamma\|_{\mathcal{B}} + \max_{\{\alpha,\beta,\gamma\}=\{a,b,c\}}\|F^\alpha\|_S\|xF^\beta\|_{\mathcal{B}}\|F^\gamma\|_{\mathcal{B}}\ ,\end{aligned} \tag{3.6.15}$$

we then using (3.6.6) to control the norm $\|xF\|_{\mathcal{B}}$. The estimate on $\|(1-\partial_{xx})^4\mathfrak{T}_\Lambda\|_S$ can be calculated similarly by replacing x with $(1-\partial_{xx})^4$. The proof is completed. □

Remark 3.6.1.
We have a Leibniz rule for $\mathcal{I}^t[f,g,h]$ and $\mathcal{N}^t[F,G,H]$, namely

$$\begin{aligned}&Z\mathcal{I}^t[f,g,h] = \mathcal{I}^t[Zf,g,h]+\mathcal{I}^t[f,Zg,h]+\mathcal{I}^t[f,g,Zh], \quad Z\in\{x,\partial_x\}\ ,\\ &Z\mathcal{N}^t[F,G,H] = \mathcal{N}^t[ZF,G,H]+\mathcal{N}^t[F,ZG,H]+\mathcal{N}^t[F,G,ZH]\ ,\quad Z\in\{x,\partial_x,\partial_y\}\ .\end{aligned} \tag{3.6.16}$$

Property (3.6.16) will be of importance in order to ensure the hypothesis of the transfer principle displayed by Lemma 3.6.2.

Bibliographie

[1] D. J. Benney and P. Saffman. Nonlinear interaction of random waves in a dispersive medium. *Proc. Roy. Soc. London, Series A, Mathematical and Physical Sciences*, 289(1418) :301–320, 1966.

[2] J. Benny and A. C. Newell. Random wave closure. *Stud. in Appl. Math.*, 48(1) :29–53, 1969.

[3] J. Bourgain. On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE. *Internat. Math. Res. Notices*, (6) :277–304, 1996.

[4] J. Bourgain. Problems in Hamiltonian PDE's. *Geom. Funct. Anal.*, (Special Volume, Part I) :32–56, 2000. GAFA 2000 (Tel Aviv, 1999).

[5] N. Burq, P. Gérard, and N. Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math.*, (126) :569–605, 2004.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness for Schrödinger equations with derivative. *SIAM J. Math. Anal.*, 33 :649–669, 2001.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Invent. Math.*, 181(1) :39–113, 2010.

[8] B. Dodson. Global well-posedness and scattering for the defocusing L^2 -critical nonlinear Schrödinger equation when $d = 2$. `Preprint arXiv:1006.1375`.

[9] P. Gérard and S. Grellier. L'équation de Szegő cubique. *Séminaire : Équations aux Dérivées Partielles. 2008–2009*, Exp. No. II, 19 p., online at `slsedp.cedram.org/`.

[10] P. Gérard and S. Grellier. On the growth of Sobolev norms for the cubic Szegő equation. *Séminaire Laurent Schwartz. 2014–2015, EDP et applications*, Exp. No. II, 20 p., online at `slsedp.cedram.org/`.

[11] P. Gérard and S. Grellier. The cubic Szegő equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(5) :761–810, 2010.

[12] P. Gérard and S. Grellier. Effective integrable dynamics for a certain nonlinear wave equation. *Anal. PDE*, 5(5) :1139–1155, 2012.

[13] P. Gérard and S. Grellier. Invariant tori for the cubic Szegő equation. *Invent. Math.*, 187(3) :707–754, 2012.

[14] P. Gérard and S. Grellier. Inverse spectral problems for compact Hankel operators. *J. Inst. Math. Jussieu*, 13(2) :273–301, 2014.

[15] P. Gérard and S. Grellier. Multiple singular values of Hankel operators. `Preprint arXiv:1402.1716`, 2014.

[16] P. Gérard and S. Grellier. An explicit formula for the cubic Szegő equation. *Trans. Amer. Math. Soc.*, 367(4) :2979–2995, 2015.

[17] P. Gérard and S. Grellier. *Hankel operators and the cubic Szegő equation, 135 p.* 2015.

[18] P. Gérard, Y. Guo, and E.S. Titi. On the radius of analyticity of solutions to the cubic Szegő equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(1) :97–108, 2015.

[19] J. Ginibre and G. Velo. Scattering theory in the energy space for a class of nonlinear Schrödinger equations. *J. Math. Pures Appl.*, 64 (9)(4) :363–401, 1985.

[20] B. Grébert and T. Kappeler. *The Defocusing NLS Equation and Its Normal Form.* Ameri. Math. Soc., 2014.

[21] B. Grébert and L. Thomann. Resonant dynamics for the quintic nonlinear Schrödinger equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(3) :455–477, 2012.

[22] M. Guardia. Growth of Sobolev norms in the cubic nonlinear Schrödinger equation with a convolution potential. *Comm. Math. Phys.*, 329(1) :405–434, 2014.

[23] M. Guardia, E. Haus, and M. Procesi. Growth of Sobolev norms for the defocusing analytic NLS on $\mathbb{T}^2$.

[24] M. Guardia and V. Kaloshin. Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation. *J. Eur. Math. Soc. (JEMS)*, 17(1) :71–149, 2015.

[25] Z. Hani. Long-time instability and unbounded Sobolev orbits for some periodic nonlinear Schrödinger equations. *Arch. Ration. Mech. Anal.*, 211(3) :929–964, 2014.

[26] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia. Modified scattering for the cubic Schrödinger equatuib on product spaces and applications. `Preprint arXiv:1311.2275v3`.

[27] Z. Hani and L. Thomann. Asymtotic behavior of the nonlinear Schrödinger equation with harmonic trapping. `Preprint arXiv:1408.6213`.

[28] P. Hartman. On completely continuous Hankel matrices. *Proc. Amer. Math. Soc.*, 9 :862–866, 1958.

[29] K. Hasselmann. On the non-linear energy transfer in a gravity-wave spectrum. i. general theory. *J. Fluid Mech.*, 12 :481–500, 1962.

[30] E. Haus and M. Procesi. Growth of Sobolev norms for the quintic NLS on $\mathbb{T}^2$. `Preprint arXiv:1405.1538`.

[31] E. Haus and L. Thomann. Dynamics on resonant clusters for the quintic non linear Schrödinger equation. *Dyn. Partial Differ. Equ.*, 10(2) :157–169, 2013.

[32] N. Hayashi, P. Naumkin, A. Shimomura, and S. Tonegawa. Modified wave operators for nonlinear Schrödinger equations in one and two dimensions. *Electron. J. Differential Equations*, pages No. 62, 16 pp. (electronic), 2004.

[33] B. B. Kadomtsev. *Plasma turbulence.* Academic Press, New York, 1965.

[34] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Proceedings of the USSR Academy of Sciences(in Russian)*, 30 :299–303, 1941.

[35] Z. Nehari. On bounded bilinear forms. *Ann. of Math.*, (65) :153–162, 1957.

[36] A. C. Newell. Wave turbulence is almost always intermittent at either small or large scales. *Stud. in Appl. Math.*, 108(1) :39–64, 2002.

[37] A. C. Newell, S. Nazarenko, and L. Biven. Wave turbulence and intermittency. *Phys. D*, 152/153 :520–550, 2001. Advances in nonlinear mathematics and science.

[38] N. K. Nikolski. *Operators, functions, and systems : an easy reading. Vol. 1*, volume 92 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz, Translated from the French by Andreas Hartmann.

[39] T. Ozawa. Long range scattering for nonlinear Schrödinger equations in one space dimension. *Comm. Math. Phys.*, 139(3) :479–493, 1991.

[40] V. Peller. Hankel operators of class $\mathfrak{S}_p$ and their applications (rational approximation, gaussian processes, the problem of majorization of operators. *Math. USSR Sb.*, 41 :443–479, 1982.

[41] V. Peller. *Hankel Operators and their applications.* Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[42] O. Pocovnicu. *Private communications.*

[43] O. Pocovnicu. Study of a nonlinear, non-dispersive, completely integrable equation and of its perturbations. Ph.D thesis à Université Paris-Sud 11, Orsay, France.

[44] O. Pocovnicu. Explicit formula for the solution of the Szegő equation on the real line and applications. *Discrete Contin. Dyn. Syst.*, 31(3) :607–649, 2011.

[45] O. Pocovnicu. Traveling waves for the cubic Szegő equation on the real line. *Anal. PDE*, 4(3) :379–404, 2011.

[46] O. Pocovnicu. First and second order approximations for a nonlinear wave equation. *J. Dynam. Differential Equations*, 25(2) :305–333, 2013.

[47] G. Staffilani. On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations. *Duke Math. J.*, 86(1) :109–142, 1997.

[48] H. Xu. The cubic Szegő with a linear perturbation. `arXiv:1508.01500`.

[49] H. Xu. Unbounded Sobolev trajectories and modified scattering theory for a wave guide nonlinear Schrödinger equation. `arXiv:1506.07350`.

[50] H. Xu. Large-time blowup for a perturbation of the cubic Szegő equation. *Anal. PDE*, 7(3) :717–731, 2014.

[51] V. E. Zakharov, V. S. L'vov, and G. Falkovich. *Kolmogorov Spectra of Turbulence*, volume 1. Springer-Verlag, Berlin, 1992.

[52] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Ž. Èksper. Teoret. Fiz.*, 61(1) :118–134, 1971.